Response to referee 1

Zhengyan Darius Shi, Dominic Else, Hart Goldman, T. Senthil

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1.

> As the authors stated, there is a discrepancy between $N > 1$ and $N = 1$. Is this a sharp discontinuity? The condition for Eq.(1.5), the main result of this paper, is $\omega^{1/3}N \ll 1$ (take $z = 3$ for example). So it seems that even when N is order 1 the incoherent part is still there.

We thank the referee for pointing out this subtlety in the interpretation of our results. We have added some clarification in the introduction, which we summarize below.

The anomaly arguments apply only in the case $N = 1$. The special feature in that case is that in each patch the boson couples to the total intra-patch density. For any $N > 1$, this is no longer the case. One presumes that non-trivial incoherent conductivity should be generic in the absence of a mechanism suppressing it, so it seems likely that there will be scale-invariant incoherent conductivity for any $N > 1$. However, the form of this incoherent conductivity at small N is not accessible using our methods. Strictly speaking, the specific frequency scaling in Eq. (1.5) is only valid for $N \gg 1$ and $(\omega/\Delta)^{1/3}N \ll 1$, where Δ is some fixed energy scale related to various coupling constants.

In any case, the key point is that the random flavor model at large N is not smoothly connected to the model at $N = 1$. In principle, there could exist some intermediate critical N_c such that the incoherent conductivity takes the form in Eq.(1.5) for $N > N_c$ and becomes trivial for $N < N_c$. We cannot rule out this possibility with our current methods.

2.

> In Sec.3.2 the authors claim that the source of low energy anomaly is due to the necessity of taking bosons into account for calculating the current vertex, where the form factor plays an important role. Eq.(3.3) is obtained by extending k to $k + A$ and expanding to linear order in A. But in cases when the form factor $g(k)$ does not contain linear in k term, does this anomaly argument fail?

Indeed, the form factor $g(\mathbf{k})$ is only a linear function of $\mathbf{k}$ in the case of a $U(1)$ gauge field. However, so long as $\nabla g(\mathbf{k})$ is not identically zero for all $\mathbf{k}$, one can still perform the expansion

$$g(\mathbf{k} + \mathbf{A}) \approx g(\mathbf{k}) + \nabla g(\mathbf{k}) \cdot \mathbf{A} + \mathcal{O}(A^2) \tag{1}$$

and obtain Eq.(3.3). After integrating by parts and approximating $\nabla_{\mathbf{k}}\psi^\dagger(\mathbf{k}+\mathbf{q}/2)\psi(\mathbf{k}-\mathbf{q}/2)$ as $\nabla_{\mathbf{k}}\rho(\mathbf{k})$, one then recovers Eq.(3.4).

3.

> From Eq.(3.3) and (3.4), the current operator from the fermions are approximated by $v_F(\theta)\tilde{n}_\theta$. Will the conclusions be altered if keeping momentum dependence in v_F in addition to the patch index? Because such complete k dependence in $v_F(k)$ is important in using Ward identity analysis.

Deep in the UV, we agree with the referee that an exact calculation of the conductivity requires the complete k-dependence of $v_F(k)$. However, upon coarse-graining to the IR fixed point theory, the UV current operator generally decomposes into a sum of scaling fields with increasing scaling dimensions

$$J_{\mathrm{UV}} = J_0 + J_1 + \dots \quad (2)$$

Fixed-point contributions to the conductivity can be obtained by keeping only the scaling field J_0 with the lowest scaling dimension. In the context of Section 3.2, we can think about Eq.(3.4) as J_0. The other scaling fields $J_{i>0}$ then correspond to additional terms in the expansion of Eq.(3.3) in powers of $\mathbf{k}-\mathbf{k}_F(\theta)$ in every patch θ. These additional terms give corrections to the incoherent conductivity in Eq.(1.5) that are subleading in ω in the small ω limit.

4.

> Is it possible to perform some numerical calculations for the patch model at $N = 1$ to justify the claim by the authors?

In the $N = 1$ case, our claim is that the boson self energy $\Pi(\boldsymbol{q} = 0, \omega) = D^{-1}(\boldsymbol{q} = 0, \omega)$ saturates to a constant value as $\omega \to 0$. This is indeed consistent with the findings of determinantal quantum Monte Carlo (DQMC) studies in [Y. Schattner, et. al., "Ising Nematic Quantum Critical Point in a Metal: A Monte Carlo Study" PRX 6, 031028 (2016)]. The residual frequency dependence in $\Pi(\boldsymbol{q} = 0, \omega)$ at small ω found in the DQMC study likely comes from irrelevant operators included in the lattice model that are outside the scope of our analysis.

As for the conductivity $\sigma(\boldsymbol{q} = 0, \omega)$, the DQMC study [Lederer, et. al., "Superconductivity and non-Fermi liquid behavior near a nematic quantum critical point", PNAS 114, 4905 (2017)] finds a finite-width Drude peak at low frequency, which appears to be in tension with the sharp δ-function Drude peak we found at finite T. We believe that the broadening of the Drude peak is due to irrelevant operators in the lattice model that degrade momentum (e.g. Umklapp scatterings are always present in the lattice model). A precise characterization of these irrelevant operators would be needed to match the DQMC results in detail. Unfortunately our anomaly arguments do not provide strong enough constraints on these effects.