Report

This manuscript analyses the steady state and dynamical properties of a Lindbladian for a bosonic field which preserves the photon-number parity. The resulting steady-state is not unique and depends on the probability (below denoted by P), that the even photon number eigenspace is initially occupied. The steady state also depends the parameter K, namely, the ratio between heating processes (incoherent emission of two energy quanta) and cooling processes (incoherent absorption of two energy quanta). The properties of the steady state are analysed as a function of the probability P and of the parameter K in terms of (i) existence of a limit cycle - identified according to criteria that the authors introduce-, and (ii) negativity of the Wigner function. The results are compared with the predictions of a classical model introduced ad hoc.

We thank the referee for taking her/his time to review our manuscript. Our responses to detailed comments can be found below. The corresponding changes in the main text are colored blue (in the colored version of the manuscript we are attaching separately.)

Note also that papers referred to in our replies here are either from the previous version of our manuscript, or appear at the end of the replies in a separate bibliography (such citations are denoted by [R1], [R2], and etc).

Requested changes

Style.

The paper is generally written with care of details and with extensive discussions. The introduction of the paper provides a nice short review of noise-induced phenomena in quantum and classical physics. The paper also contains an extensive analytical analysis in the appendices. There are few typos (e.g., caption Fig. 1: "embbed" / after Eq. (23) in the appendices: "subtituting"). Suggestion: In Eq. (12). It would be useful to comment in the main text that x and y are the bosonic field's quadratures. Remark: Sentence before Eq. (96): P__0 and P__1 are used in an argumentation but defined only later on.

We thank the referee for carefully reading the manuscript. We have fixed the typos in the revised manuscript.

Questions to the authors.

1) Thermal bath. In order to relate their study to noise-induced resonances, the authors refer to their model, Eq. (1), as a nonlinear damped oscillator coupled to a heat bath. In general, while Eq. (1) has an interesting theoretical justification on its own, I do not see how one can associate a temperature with it. In fact, Eq. (2) is generally not a thermal state. It is peculiar, that Eq. (2) does not even asymptotically reach a thermal state for K->1, which, according to the authors, shall be the case of very large temperatures. The authors shall provide an accurate justification of the use of the concept of temperature in their model (see also item 6 of this list)

We thank the referee from raising a possible point of confusion. It is important to mention that we only refer to "temperature" as the temperature of the atomic bath which is traced out in master equation (1) (see the opening paragraph of Sec. II of our paper). The atomic ensemble in the bath is thus in a thermal state at temperature T [see (27) in the Supplementary Material of Ref. [57] from our paper for a detailed discussion]. Therefore, the oscillator is not thermalized to temperature T, as pointed out by the referee. However, the parameter K increases with the temperature (and hence thermal noise) of the bath, so $K \longrightarrow 1$ for the oscillator is equivalent to $T \longrightarrow \infty$ for the atomic bath.

We have also revised our references to "the temperature" in the Introduction as being the temperature of the bath so that this point is made clear at least twice (once in Sec. I and once more in Sec. II).

2) Steady state I. Equation (2) does not contain off-diagonal elements between the even photon number and the odd photon number eigenspaces. The authors make reference to previous literature in order to motivate this ansatz. They also show that for K=0 this ansatz is not valid: In fact, in this limit there is a perfectly decoupled subspace spanned by the photon number states |0> and |1>. Since Eq. (2) shall hold for K>0, is the limit K->0 different from the case K=0? And in general, how shall one understand the limit of small K in the phase diagrams of Fig. 2 and 3? For instance: assume an initlal state that is a coherent superposition of 0 and 1 with equal probability: How do the off-diagonal elements between the two subspaces behave at long times for small K?

Yes, the $K = 0$ case is qualitatively different from the $K = 0$ case. For $K = 0$, one has an extra conserved quantity for the off-diagonal coherence on top of parity conservation [see (3.14) of Ref. [131] in our paper for details], while the $K > 0$ case only has parity conservation. Hence, the phases plotted on the left-edge of Fig. 2 in the old manuscript is strictly speaking for the $K \longrightarrow 0^+$ limit. This marks a thermal noise-induced transition from $K = 0$ to the various phases shown in Figs. 2 and 3 of the old manuscript depending on the initial parity.

As per the referee's question, if we initialize in the coherent superposition of the even and odd parity states, the off-diagonal coherence will decay to zero on a timescale of K^{-1}. Hence, the parity is conserved in the steady state only if $K = 0$.

We have added a few sentences to clarify this point in the revised text. See the end of the first paragraph on p6.

3) Steady state II. In connection with the previous item. Equation (109) in the appendices shall justify the form of Eq. (2). Is this formula also applicable to the steady state of the master equation with K=0?

Yes, (109) from our Supplementary Material is applicable even if $K = 0$, but in that case there will be an additional conserved quantity for the coherence as mentioned in the previous answer, so we have $D = 3$ instead.

4) Phase diagram I. Which physical observable shall distinguish Phase II from Phase III?

Phases II and III can be distinguished by measuring the parity of the state $\widehat{\Pi} = (-1)^{\hat{a}^\dagger \hat{a}}$, which is related to the Wigner negativity using $W(0) = 2\mathrm{Tr}(\widehat{\Pi}\rho)/\pi$. The expectation value of parity can be obtained in experiments via non-demolition measurements [R1].

5) Phase diagram II. Some features of the phase diagram depend on K. However, one important result of

this paper (Phase III) seems to solely depend on the initial state, and specifically on P. In view of this result, the claim of quantum noise-induced resonance is questionable.

Our quantum noise-induced transitions refer to (i) the formation of various phases I to III from $K = 0$ to $K > 0$ as discussed in a previous answer; and (ii) the stochastic Hopf bifurcation between phases I and II which occurs for a critical K. The justification for "quantum noise-induced" transitions comes from the comparison with the analogous classical noisy model which shows no such behaviour in all parameter regimes. We have added a few sentences on the second paragraph of p2 to explain this.

As the referee has noted, once the state is in Phase III, no amount of thermal noise can remove the Wigner negativity. This is equivalent to the conservation of parity and is not related to the stochastic Hopf bifurcation.

6) Classical limit. The authors argue that Equation (9) shall provide a classical benchmark to their model, Eq. (1). The justification of this statement is not provided. The authors comment in the text about the difficulty of taking the classical limit of Eq. (1). As an alternative, they could consider the original model (emitters + photon field) and derive the Fokker-Planck equation in the semiclassical limit. In this case they would find a classical model with which they can consistently benchmark Eq. (1) (I suspect that they will have to take into account photon processes that break the symmetry of Eq. (1)). and also possibly identify the order of magnitude of K for which Eq. (1) is no longer valid.

As mentioned in the page 4 of the main text "Although our classical model … completely deterministic", the usual method of system size expansion for obtaining semiclassical Fokker-Planck equation of motion fails for our model due to the presence of multiplicative noise which scales with the system size. Other equivalent ways of obtaining semiclassical limits like truncated Wigner approximation and mean-field approximation will fail for the same reason. Fundamentally, our model lacks a proper classical limit unlike quantum models with additive noise. [See the added sentences in red on p5 (also a response to referee 1), at the end of section II.]

Hence, for comparison purposes, we construct the closest classical approximation to the noisy dynamics given in (10). The classical noise model is constructed from the quantum Langevin equation by allowing the bosonic operators to commute and replacing them with c-numbers. This is an alternative method to find the semiclassical limit. This is not to be misconstrued as the physical classical limit of a quantum system, as we have mentioned in page 4. Our approach amounts to allowing the bosonic operators to commute and replacing them with c-numbers. Therefore, the differences in behaviour between the quantum model (1) and the analogous classical model (10) can be attributed to the non-commuting nature of quantum operators, which is a genuine quantum effect.

We remark that for additive noise, this procedure is equivalent to a semiclassical approximation but this is not true for multiplicative noise in general (see the second paragraph in Sec. 4 on page 5 of Ref. [R2], also cited as Ref. [63] in our paper). We also refer to our reply to the second point raised in the first referee report.

References

[R1] L. Sun, A. Petrenko, Z. Leghtas, B. Vlastakis, G. Kirchmair, K. M. Sliwa, A. Narla, M. Hatridge, S. Shankar, J. Blumoff, L. Frunzio, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, Tracking photon jumps with repeated quantum non-demolition parity measurements, Nature **511**, 444 (2014).

[R2] T. Weiss, A. Kronwald, and F. Marquardt, Noise-induced transitions in optomechanical synchronization, New J. Phys. **18**, 013043 (2016).