Response to referees and list of changes

Dear Referee,

We carefully address, point by point, the referee's recommendations and comments in relation to the manuscript scipost_202212_00060v1 entitled *Mixed permutation symmetry quantum phase transitions of critical three-level atom models*, by A. Mayorgas, J. Guerrero, and M. Calixto, which is submitted to SciPost Physics Proceedings for publication. We have also included the references suggested by the referee. As a result, the length of the article slightly exceeds 8 pages. We hope that this is not a problem.

ANSWER TO REFEREE

Point 1) "page 2, second line: What is more "exotic" about N particles distributed in L levels?...".

Answer: This point has been clarified and a citation has been included in the manuscript.

Point 2) "page 2, last paragraph of section 1, "we classify the Hamiltonian spectrum and examine..." is in fact false. . . ".

Answer: This point has been clarified in the manuscript, we focus on a simplified version of the Hamiltonian for L=3 levels.

Point 3) "The statement "The symmetry will be spontaneously broken in the thermodynamics limit" should be supported by a citation. . . ".

Answer: How the parity symmetry is spontaneously broken is explained in the last paragraph on Section III. We have included a reference studying this phenomena in the LMG U(2) model, first paragraph of page 2.

Point 4) "I could go on: the current writing is very sloppy and very unclear. . . ".

Answer: The unclear concept of "quasi-spin operator" has been replaced by u(n) generators. Unnecessary and colloquial comments have been removed.

Point 5) "The constant referencing to Ref[5] is inappropriate and misleading. . . ".

Answer: The constant referencing to Ref[5] (now Ref[6] in the new version of the manuscript) has been replaced by the citations suggested.

Point 6) "For U(3) irreps labeled by a one-rowed Young diagram, the expression…".

Answer: A correction has been included in the first paragraph on Section III. Only the highest weight vector of the fully symmetric U(3) irrep is U(2) invariant. The highest weight vector in the equation above depends on the U(3) irrep chosen, therefore, for a generic irrep, the U(3) coherent state is labeled by the three complex coordinates α, β, γ and the irrep label h. This parameterization is due to the Bruhat decomposition, already cited and related to the flag manifolds in the second paragraph on page 3.

Point 7) "The parametrization of U(3) elements given in Eq.(3) is very unusual…".

Answer: All the references suggested have been considered and cited in the second paragraph on page 3. The first version of this paper only included a reduced number of references due to the length limitation of 8 pages, but it has increased to 9 pages in the current version with the new references.

Point 8) "There are others and it would therefore seems entirely appropriate for the authors to clarify the benefits of using such an unusual representation…".

Answer: The parametrization is chosen for convenience, what is also explained in the second paragraph on page 3 of the new version.

Point 9) "Beyond the somewhat cosmetic changes mentioned above…".

Answer: The Figure 1 shows the susceptibility of the numerical eigenstates of the LMG U(3) Hamiltonian in different representations vs the control parameter λ. To perform this numerical diagonalization, we calculate the matrix form of the Hamiltonian for a given unirrep using the Gelfand-Tsetlin method. Therefore, the numerical diagonalization does not involve coherent states, that is why there is no reference to the parameters α, β, γ in this figure.

In Figure 2, we plot the lowest-energy density defined in the eq(6) for different unirreps (labeled by μ) vs the control parameter λ. The lowest-energy density is calculated by minimizing the energy surface (eq.(5)) in the phase space (α, β, γ), therefore, the plot is independent of these parameters.

Point 10) "The authors cite Ref[4] to justify the use of coherent states as trial states…".

Answer: We have included the reference "R. Gilmore, Catastrophe Theory for Scientists and Engineers (Wiley, New York, 1981)" where it is explained the general procedure of using coherent states as variational states.
The last part on Section II is devoted to calculating the exact/numerical eigenstates of the LMG U(3) Hamiltonian and give some QPT precursors. The exact eigenvalues have been calculated for $N = 50$ and even for $N = 100$ as shown in Figure 1(a). When increasing the number of particles N (higher dimension in the symmetric unirrep), Figure 1(a) tells that the maximum of the susceptibility is attained closer to $\lambda = 0.5$, foreseeing the critical $\lambda^{(0)}$ which gives the variational ansatz. That is how the exact eigenstates are compared with the variational ansatz. In the case of the Figure 1(b), we present different unirreps with the same number of particles N.

Point 11) "The N-fold tensor product of the fundamental representation…".

Answer: The multiple copies of the same representation have the same energy spectrum and eigenstates, that is why they have not been mentioned in this work. In other words, the Hamiltonian matrix in the whole Hilbert space (N-fold tensor product of fundamental U(3) representation) can be block diagonalized, where each block correspond to an irrep, and the multiple copies of an irrep have identical blocks. Hence the energy spectrum and fidelity are the same for all these multiple copies, so the Figures 1 and 2 do not include repeated irreps.
When taking the thermodynamic limit, it is convenient to relabel the U(3) irrep $[h_1, h_2, h_3]$ by the continuous parameters μ, ν, as explained in the first paragraph on page 5. Since the Hamiltonian in the eq.(2) is defined as an energy density (intensive quantity), we can easily take its CS expectation value thermodynamic limit as in the eq.(5) (energy surface). The permutation symmetry of an unirrep does not have any special meaning in the $N \to \infty$ limit, and it does not make sense to compute its states in this limit.

We thank the referee for their comments and recommendations which helped to improve the article. I hope that the explanations provided and the changes made to the manuscript make it suitable for publication in SciPost Physics Proceedings.

Looking forward to hearing from you.
Yours sincerely,
The Authors