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To Referee 1 of paper *scipost_202212_00073v1*

SUBJECT: Reply on Report 1

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t' Hooft lines of ADE type and Topological Quivers

Dear referee,
Thank you for reading our paper and for the useful comments and suggestions regarding the consideration of our paper.
Please find below answers to the questions of your report as well as the modifications that we have made in the new version of the paper.

Best Regards

Answer to report and List of modifications

N.B:
In the beginning of this document, we would like to draw the attention of the referee to the following points
∗ *The answers to the referee's questions are listed in the same order of the initial report.*
∗ *The recommendations and corrections are implemented in the second version of the paper.*
∗ *An appendix is added (Pages 68-69).*

- **Concerning the construction of the topological quiver:**

  – **Answer to question:**
    Indeed, as noticed in the referee's report, a topological quiver is characterized by $(G; \boldsymbol{R}; \mu)$ where the minuscule coweight μ decomposes the Lie algebra g of the gauge symmetry G as

$$g = n_+ \oplus l_\mu \oplus n_-$$

with l_μ being the Levi subalgebra of g. Because of this splitting, the representation $\boldsymbol{R}$ of the gauge symmetry G is decomposed into p representations $\boldsymbol{R}_{m_i}$ (irreps) of l_μ like,

$$\boldsymbol{R} = \sum_{i=1}^{p} \boldsymbol{R}_{m_i} = \sum_{i=1}^{p} \boldsymbol{R}_i$$

This is due to the adjoint action of μ on subspaces characterized by projectors $\Pi_1, ..., \Pi_p$ as follows

$$\mu = \sum_{i=1}^{p} m_i \Pi_i, \qquad \sum_{i=1}^{p} \Pi_i = \mathbf{1}_{\boldsymbol{R}}, \qquad Tr_{\boldsymbol{R}}\left(\mu\right) = \sum m_i Tr\left(\Pi_i\right) = 0$$

The m_i is the charge of $\boldsymbol{R}_i$ with respect to the $so(2)$ of μ appearing in the Levi subalgebra as for the example of the decomposition of $g = sl(N)$ under the k-th coweight $e_k^\vee \equiv \mu_k$ yielding $l_{\mu_k} = sl(k) + so(2) + sl(N-k)$.

To such data $(G; \boldsymbol{R}; \mu)$, we associate a quiver having p nodes $\mathcal{N}_i$ and $2(p-1)$ oriented links $L_{i\to j}$ and $L_{j\to i}$ connecting $\mathcal{N}_i$ and $\mathcal{N}_j$. Each $\mathcal{N}_i$ is characterized by the data $(\boldsymbol{R}_i, m_i) \equiv \boldsymbol{R}_{m_i}$ where the charge is denoted as a subscript of the irrep. Because of the conservation of the Levi-charge running in the quiver, the links also carry Levi charges. Actually, the branching rules give

$$m_i - m_{i+1} = \pm 1, \qquad m_i - m_j = \pm k$$

where $k = 1, ..., p-1$ is an integer. The links are therefore given by polynomials in the Weyl oscillators $\mathbf{b} = (b^\alpha)$ and $\mathbf{c} = (c_\alpha)$ which carry charges ∓ 1 with respect to μ as required by the nilpotent $\mathbf{n}_\pm$ as shown by

$$e^X = e^{b^\alpha_{(-1)} X_{\alpha(+1)}}, \qquad e^Y = e^{c_{\alpha(+1)} Y^\alpha_{(-1)}}$$

So, given two nodes $\mathcal{N}_i$ and $\mathcal{N}_j$ with Levi-charges verifying $m_j - m_i = k$, the link $L_{i\to j}$, carrying charge $+k$, is given by a polynomial generated by monomials of the form $\mathbf{c}^{k+l}\mathbf{b}^l$ with leading element given by $\mathbf{c}^k$. Similarly, the link $L_{j\to i}$, carrying a Levi charge $-k$ is generated by $\mathbf{b}^{k+l}\mathbf{c}^l$ with leading monomial $\mathbf{b}^k$. The arrows are added appropriately to conserve the proper circulation of the Levi charge in the quiver.
Regarding the contributions of these nodes and links in the Lax operators $\mathcal{L}$, they are determined by using the resolution of the identity operator ($\sum \Pi_i = \mathbf{1}_{\boldsymbol{R}}$) as

$$\begin{aligned}\mathcal{L} &= \mathbf{1}_{\boldsymbol{R}}\mathcal{L}\mathbf{1}_{\boldsymbol{R}} \\ &= \sum_i \Pi_i\mathcal{L}\Pi_i + \sum_{i>j} \Pi_i\mathcal{L}\Pi_j + \sum_{i<j} \Pi_i\mathcal{L}\Pi_j\end{aligned}$$

where the nodes $\mathcal{N}_i$ are associated to the diagonal sub-blocks $\Pi_i\mathcal{L}\Pi_i$ and the links to the off-diagonal blocks $L_{ij} = \Pi_i\mathcal{L}\Pi_j$.
For the example of

$$g = sl(N); \qquad \boldsymbol{R} = \boldsymbol{N}; \qquad \mu = \frac{1}{N}\Pi_1 + (1 - \frac{1}{N})\Pi_2$$

the Levi subalgebra is given by $sl(N-1) \oplus so\,(2)$ and the associated quiver has two nodes $\mathcal{N}_1$ and $\mathcal{N}_2$; and two oriented links $L_{1\to 2}$ and $L_{2\to 1}$ carrying Levi charges ± 1 and written in terms of the two oscillators $(\mathbf{b}, \mathbf{c})$ as follows

$g = sl(N)$	$sl(N-1) \oplus so\,(2) \oplus \mathbf{n}_+ \oplus \mathbf{n}_-$			
quiver	Node $\mathcal{N}_1$	Node $\mathcal{N}_2$	Link $L_{1\to 2}$	Link $L_{2\to 1}$
$\boldsymbol{R} = \boldsymbol{N}$	Irrep. $\underline{\mathbf{1}}$	Irrep. $\underline{\mathbf{N-1}}$	$\mathbf{b}$	$\mathbf{c}$
Levi charge m_i	$\mathrm{m}_1 = 1 - \frac{1}{N}$	$\mathrm{m}_1 = -\frac{1}{N}$	-1	$+1$
Lax op. element	$\Pi_1\mathcal{L}\Pi_1$	$\Pi_2\mathcal{L}\Pi_2$	$\Pi_1\mathcal{L}\Pi_2$	$\Pi_2\mathcal{L}\Pi_1$

– **Modification in the revised paper**
To shed more light on this construction in the paper, we reformulated the subsection 3.1 (**see pages 19-20**).

- **Concerning the possibility of generating Lax operators from topological quivers:**
We thank you for pointing out to the importance of explaining this point.

  – **Answer to question**
  In fact, a quiver description does allow to obtain the expression of the corresponding Lax matrix $\mathcal{L}$ such that :

  * The number p of quiver nodes $\mathcal{N}_i$ gives (**1**) the number of sub-blocks $\Pi_i\mathcal{L}\Pi_i$ of the L- matrix; and (**2**) defines (**i**) the projectors Π_i and then the subspaces appearing in the decomposition of $\boldsymbol{R} = \sum_{i=1}^{p} \boldsymbol{R}_i$ with the dimensions

$d_i = \dim \boldsymbol{R}_i$. (**ii**) The Levi charges m_i are directly read from $\mathcal{N}_i$ and the conservation law given by the trace $Tr\boldsymbol{R} = \sum_{i=1}^{p} m_i d_i$ with d_i also given by $Tr\,(\Pi_i)$. The diagonal sub-blocks should also follow the order of the nodes in the quiver to conserve the property $m_i - m_{i+1} = \pm 1$.

* The contributions of the diagonal sub-blocks $\Pi_i \mathcal{L} \Pi_i$ with charges m_i (i.e: the node $\mathcal{N}_i$ with $1 \leq i \leq p$) are given by polynomials in $\mathbf{:b}^l\mathbf{c}^l\mathbf{:}$ reading explicitly as follows

$$\begin{aligned}
\mathcal{L}_{11} &= z^{m_1} + z^{m_2}\mathbf{bc} + ... + z^{m_p}\mathbf{b}^{p-1}\mathbf{c}^{p-1} \\
&\vdots \\
\mathcal{L}_{ii} &= z^{m_i} + ... + z^{m_p}\mathbf{b}^{p-i}\mathbf{c}^{p-i} \\
&\vdots \\
\mathcal{L}_{pp} &= z^{m_p}
\end{aligned}$$

where we omitted possible coefficients of these terms for simplicity. Notice that monomials $\mathbf{:b}^l\mathbf{c}^l\mathbf{:}$ carry no Levi charge as $\mathbf{b}$ and $\mathbf{c}$ have opposite charges.

* The contribution of the links $L_{ij} = \Pi_i \mathcal{L} \Pi_j$ is given by polynomials in the Weyl oscillators $\mathbf{b}$ and $\mathbf{c}$ with charges $\pm 1, ..., \pm(p-1)$. Typical forms of these polynomials are given by

$$\begin{aligned}
\mathcal{L}_{i,i+1} &= z^{m_{i+1}}\mathbf{b} + ... + z^{m_p}\mathbf{b}^{p-i}\mathbf{c}^{p-(i+1)} \\
\mathcal{L}_{i+1,i} &= z^{m_{i+1}}\mathbf{c} + ... + z^{m_p}\mathbf{c}^{p-i}\mathbf{b}^{p-(i+1)} \\
\\
\mathcal{L}_{i,j} &= z^{m_j}\mathbf{b}^{j-i} + ... + z^{m_p}\mathbf{b}^{p-i}\mathbf{c}^{p-j} \qquad ; \quad j > i \\
\mathcal{L}_{j,i} &= z^{m_j}\mathbf{c}^{j-i} + ... + z^{m_p}\mathbf{c}^{p-i}\mathbf{b}^{p-j} \qquad ; \quad j > i
\end{aligned}$$

However, since the phase space coordinates $\mathbf{b}$ and $\mathbf{c}$ can be given by vectors or tensors in general, depending on the realisation of X and Y, these terms could be accompanied, by metric tensors to homogenize the indices inside each block.

– **Modification in paper**
In order to include this material in the paper, we added an appendix where we described these general steps for the construction of the Lax matrix form a gauge quiver. We also gave an example by building the L- matrix corresponding to the gauge quiver $Q_{\boldsymbol{adj}}^{\mu_k}$ of sl_N (**see pages 68-69**).

- **Concerning the link with ADE gauge quivers in Costello-Gaiotto-Yagi's paper:**
We thank you for this interesting suggestion.

– **Answer to question:**
We are currently studying the possible connection between these gauge quivers and ours.

- **Modification in paper:**
  Meanwhile, we pointed to this in the conclusion section (**page 63**) for motivating future research aspects in this direction.

- **Concerning the L-operators for representations not lifting to the Yangian:**
  We thank you for this remark.

  - **Answer**:
    In our paper, we constructed several gauge quiver diagrams corresponding to minuscule coweights μ and different representations $\boldsymbol{R}$ of a gauge symmetry G. These correspond to the coupling of a Wilson line with electric charge $\boldsymbol{R}$, and a 't Hooft line with magnetic charge μ in the 4D CS theory. They describe the behaviour of the gauge configuration in the presence of disorder operators and visualise the effect of the Dirac like singularity in such gauge theory. The corresponding L- operators define the phase space of the 't Hooft line tH^{μ} eq(2.8). However, by the 4D CS/ Integrability correspondence, the L-operators associated to these topological quivers are nothing but Lax operators of a dual XXX spin chain. These operators verify, along with the quantum R-matrix, the RLL equation of integrability requiring a Yangian realisation.
    As you mentioned, the adjoint representations of SO_{2N}, E_6 and E_7 do not lift to the corresponding Yangian and therefore the associated system can't be quantumly treated. We choose to keep these examples as they may be interpreted differently, as semi-classical Lax operators, or as 't Hooft lines.
  - **Modification in paper:**
    To point to this, we added a sentence in **page 19** and in the conclusion section, **page 63**.

- **Concerning references for Lax matrices:**
  We added the corresponding references next to equations (3.12), (3.20), (4.25), (5.60), (6.33), (7.25).

- **Minor Comments**

  - The equation of motion of 4D CS is corrected in eq(2.4).
  - eq(2.14) corrected.
  - After eq(2.24), we added "sl_1 refers to $\mathbb{C}$".
  - In table (2.26) and in the rest of the text, the $\bar{\mathbf{1}}_{\frac{1}{N}}$ is replaced by $\mathbf{1}_{\frac{1}{N}}$, and the tensor product is explicitly written.
  - The combination $Y\rho_{\bar{1}}$ should indeed be $\rho_{\bar{1}}Y$; the expression of this L-operator is corrected in eq(3.10) in agreement with eq(2.41).

- Arrows fixed in Figure 11.

- **Other Modifications**
  - Subsection 3.2 was refined to avoid repetition with 3.1.
  - Notations of Lax matrices and quivers were homogenized through the paper like $\mathcal{L}^{\mu}_{\boldsymbol{R}}$ and $\mathrm{Q}^{\mu}_{\boldsymbol{R}}$.

Best regards,

Youssra Boujakhrout, for the authors