Reply to Report 2 by Gabriel Lemarié

Strengths:

1- The authors consider 5 types of random graphs to assess the universality of their results.

2- The authors determine the critical value of the disorder strength independently of the finite-size scaling procedure. This is usually a major source of uncertainty in the determination of the critical exponent, so this is particularly interesting.

3- The authors propose an alternative scenario for the existence of a finite-range in system size non-ergodic regime in the delocalized phase, associated to the existence of a new length scale ξ with a critical exponent $\nu = 1$.

We thank Dr. Lemarié for the careful reading of our work and the positive assessment of its significance and originality.

Weaknesses:

The numerical demonstration of the new critical exponent $\nu = 1$ does not seem to me convincing. I propose an alternative analysis of the data of the authors which shows that they are perfectly compatible with $\nu = 1/2$. These two approaches should be compared quantitatively.

Below, we address the critical remarks raised in the Report of Gabriel Lemarié showing a quantitative comparison between our approach with the critical exponent $\nu = 1$ and the approach with $\nu = 1/2$.

The authors mainly consider a very popular, although rather imprecise, observable of localization: the average gap ratio $\overline{r}$

We decided to employ the average gap ratio $\overline{r}$ in our analysis of the Anderson transition on random graphs to be able to quantitatively compare the finite system size drifts at the delocalization/localization crossover in this quite well controlled setting (with a known critical disorder strength W_C) to earlier investigations of the MBL transition. The $\overline{r}$ parameter behaves like any other spectral indicator, measuring the probability that level is repelled from the neighboring ones, it does not show any pathological scaling that we know of.

$$L_T = a_0 + a_1 (W_C - W)^{-\nu} \tag{1}$$

I would suggest that the authors test the behavior (1) with their data for the other models they have considered (I have done that for RRG D=4 and it works very well also) and compare the goodness of fit with the behavior they propose $L_T(W) \sim (W_C - W)^{-1}$. Moreover, it seems to me that their estimation of W_∞^T depends crucially on the range of system sizes where they make a linear fit of $W^T(L) - W_\infty^T$ as a function of $1/L$. Could the authors quantify that uncertainty?

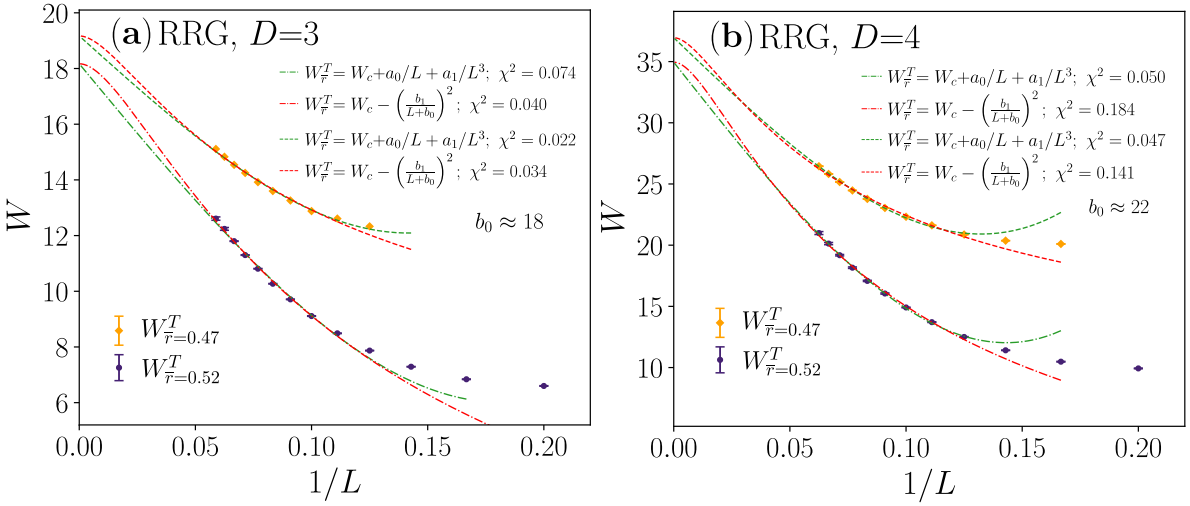

Figure 1: Disorder strength $W^T(L)$ as function of $1/L$ where L is the system size is for RRG with $D = 3, 4$ respectively in panels (**a**) and (**b**). For better visibility, the data for $W^T_{\bar{r}=0.47}$ are shifted upwards by 1 (2) for $D = 3$ ($D = 4$). The red dashed lines show the fits with of (2) and the green dashed lines correspond to the fits of (3).

The requested comparison is presented in Fig. 1. For presentation purposes, we invert (1) which leads us to

$$W^T_{\bar{r}} = W_C - \left(\frac{b_1}{L + b_0}\right)^2. \tag{2}$$

This behavior of $W^T_{\bar{r}}$ can be compared with the $1/L$ behavior consistent with $\nu = 1$. To compare formulas which have the same number of fitting parameters equal to 2 (b_0 and b_1 in (2)), we include also the first sub-leading term not written explicitly in the Eq. (12) in our manuscript, which is $\sim L^{-\omega-1/\nu} \sim L^{-3}$. This results in the following formula

$$W^T_{\bar{r}} = W_C + a_0/L + a_1/L^3, \tag{3}$$

with 2 free parameters a_0 and a_1.

To quantitatively compare the hypotheses (2) and (3), we calculate

$$\chi^2 = \sum_i \left(W^T_{\bar{r}}(l_i) - f(l_i)\right)^2, \tag{4}$$

where the sum extends over system sizes $l_i \geq 9$ for $D = 3$ and $l_i \geq 8$ for $D = 4$. Moreover, to check the robustness of the results, we consider two values of $p_{\bar{r}}$ and study both $W^T_{\bar{r}=0.52}$ and $W^T_{\bar{r}=0.47}$. The values of χ^2 displayed in Fig. 1 show that both functions (2) and (3) reproduce the behavior of $W^T_{\bar{r}}$ with comparable accuracy. Furthermore, the crossover to the asymptotic $\sim 1/L$ behavior of (3) which correctly reproduces the value of W_C occurs already for system sizes $L \approx 12$. In contrast, the coefficient b_0 in (2) is of the order of the largest system size available (the value of b_0 is given in Fig. 1), which implies that the crossover to the asymptotic behavior $1/L^2$ consistent with $\nu = 1/2$ occurs only at system sizes $L \gg b_0$, beyond the reach of present numerical methods.

As pointed out in the Report, our estimate of W_∞^T does depend on the range of system sizes taken into account in a fit $W_{\overline{r}}^T = W_\infty^T + a_0/L$. The values of W_∞^T listed in our manuscript were obtained for the 5 largest system sizes available (with the exception of RRG $D = 4$ for which we used 3 data points at the largest L). The obtained values of W_∞^T are close to W_C, in accordance with our hypothesis of $\nu = 1$. When we include smaller system sizes and try to fit $W_{\overline{r}}^T = W_\infty^T + a_0/L$, we obtain smaller values of W_∞^T. For instance, using points with $L \geq 10$ ($L \geq 8$) we get $W_\infty^{\prime T} = 17.5$ ($W_\infty^{\prime T} = 16.6$) for RRG with $D = 3$. Such an extension of the fitting interval is, however, not well justified due to the obvious curvature (on the $1/L$ axis) of the data for smaller L, which shows that the sub-leading terms in the scaling play an important role at smaller system sizes.

All in all, we believe that the above comparison shows that the behavior of $W_{\overline{r}}^T$ at the Anderson transition on random graphs is described with similar accuracy by fitting forms consistent both with $\nu = 1/2$ and $\nu = 1$. To highlight this point in our work, we included Fig. 1 in the Appendix of our manuscript. Data for larger sizes (say up to $L_m = 20 - 24$ for RRG with $D = 3$) could be used to check whether the value of W_∞^T starts to overestimate the value of W_C as we increase L_m beyond $L_m = 17$. Such a behavior would indicate that there is another change of the curvature (on $1/L$ scale) of $W_{\overline{r}}^T$ data at even larger system sizes, which could suggest that $1/L$ behavior is not the asymptotic one. However, we presently have no numerical resources to check this hypothesis (we note that obtaining numerical results for $L = 20$ for SWN or URG is significantly less demanding than for RRG with $D = 3$).

> One important new point of the argumentation of the authors is that ω should be equal to 2. In the insets of figure 8, they show that the numerical data seem to follow this trend at sufficiently large system sizes. The value of $\omega = 2$, together with $\nu = 1$, is crucial to explain the observation of an "effective" critical exponent $\nu_{eff} = 1/2$ for the crossover to delocalization observed in several references, see e.g. [30] and arXiv:1810.07545. I say effective because, according to the analysis of the authors, the true critical exponent is $\nu = 1$. I don't understand the claim of the authors that the data follow the trend with $\omega = 2$, even at large system sizes. In Fig. 2, I plot directly $\ln(\overline{r}(W_C) - r_P)$ as a function of $\ln L$. I observe clearly a linear behavior, consistent with Eq. (4) with however $\omega = 2$ and not universal. In particular, I do not observe a different trend of the data at "large" system sizes as compared to small ones. Why do the authors plot in the insets of Fig. 8 their data as $\overline{r}$ as a function of $1/L^2$?

We believe that our results for $\overline{r}(W_C) - \overline{r}_P$ do suggest universal behavior valid for different types of random graphs. Moreover, this behavior seems to crossover to a L^{-2} scaling at the largest system sizes available. To illustrate this point, we plot $\overline{r}(W_C) - \overline{r}_P$ as function of system size $\overline{L}$ on a log-log scale, see Fig. 2. The value $\overline{r}(W_C)$ is extracted as the value of a cubic spline that interpolates our data for $\overline{r}(W)$ taken exactly at $W = W_C$ (this leads only to minor shifts of the results shown in the Report; we note that we supplemented our results for $D = 4$ with data for larger disorder strengths at $L \leq 10$, see Fig. 14(**a**) in the manuscript). Importantly, the curvature of $\overline{r}(W_C) - \overline{r}_P$ for URG and SWN is visible on the log-log scale when the results are plotted as a function of $\overline{L}$, which we believe is the relevant variable that describes the size of the system. As pointed out in the report, the data at intermediate system sizes are described by $L^{-\alpha}$ dependence with $1 < \alpha < 2$. However, the power α increases with increasing system size and is close to 2 for the largest system sizes available, which is especially well pronounced for RRG with $D = 3$. Thus, we believe that plotting

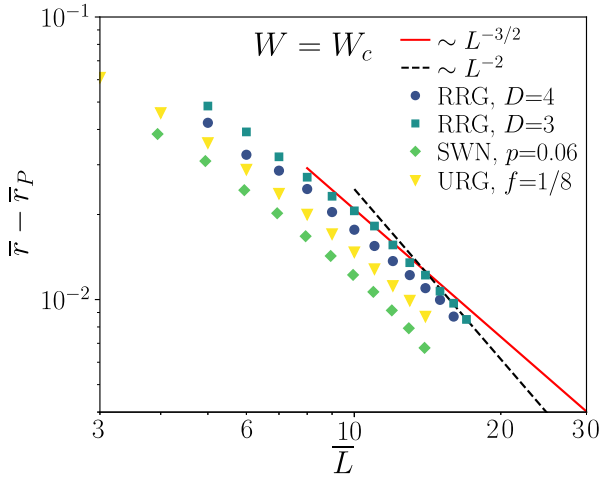

Figure 2: The value of $\overline{r} - \overline{r}_P$ at the critical point $W = W_C$ for Anderson models on random graphs. The number of vertices with degree bigger or equal to 3 is given as $2^{\overline{L}}$, i.e. for RRG with $D = 3, 4$ we have $L = \overline{L}$; for URG we have $\overline{L} = L + \log_2(2p)$; for SWN, $\overline{L} = L + \log_2 f$ (consistently with the manuscript). The solid and dashed lines correspond to $\overline{L}^{-3/2}$ and $\overline{L}^{-2}$ behaviors. Errorbars are smaller than data points.

the data as function of $1/L^2$, as done in Fig. 8 of the manuscript, is justified. However, in order to give a complete account of the data, we also include Fig. 2 in the Appendix to the manuscript.

> The data in the scaling plots reach at $W \to W_C$ the value $\overline{r}_P$. However, the authors have also data for $W > W_C$, in the localized regime. How do these data scale? They have values lower than $\overline{r}_P$? How to understand that? The authors suggest a modified scaling assumption, Eq. (10) to describe this regime, but how do they justify its form and how precisely this works in the localized regime?

Our scaling assumption predicts that $\overline{r}$ is always larger than the Poisson value $\overline{r}_P$. This is enforced by the forms of the functions $f(x)$, which vanishes at $x > 0$, and $f_1(x)$, which decays exponentially with L for $x > 0$. To keep the scaling analysis more restrained, we consider only the first non-trivial term in $f_1(x)$ at $x < 0$, namely $f_1(x) = A$. This is justified in the vicinity of the critical point. In particular, in order to better describe the data at $W > W_C$, we would have to include terms in $f_1(x)$ that would ensure its exponential decay at $x > 0$. This is implied by the behavior of $\overline{r} - \overline{r}_P$ which decays exponentially with system size L at $W > W_C$, as shown in Fig. 3. The exponential decay is clearly visible at $W \geq 22$ for the available system sizes, but we believe that it occurs at all $W > W_C$ for sufficiently large L. Presently, we do not have a precise theoretical argument that would indicate the exponential decay of $\overline{r} - \overline{r}_P$ at $W > W_C$. We believe that such an argument can be provided for other measures of localization in the system, which we, however, leave for future work. In the present work we concentrate on the scaling collapses for $x < 0$. For that reason we do not include in our finite size scaling analysis terms which describe the exponential decay of $\overline{r} - \overline{r}_P$ in the localized phase.

> Another question is why the authors consider a limited range of system sizes in their scaling analysis? They have for the SWN with $p = 0.06$ data for $7 \leq L \leq 16$. Could

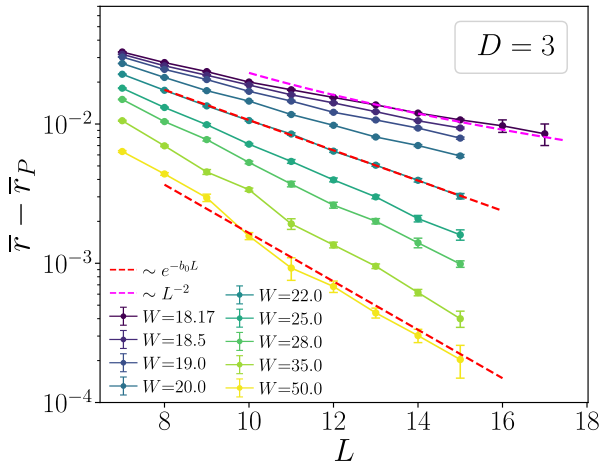

Figure 3: The value of $\overline{r} - \overline{r}_P$ at $W > W_C$ for Anderson models on RRG with $D = 3$ as a function of system size L. The dashed magenta line shows asymptotic behavior $\sim L^{-2}$ at the critical point $W = W_C$. The dashed red lines show an exponential decay $\sim e^{-b_0 L}$ where b_0 is a disorder strength dependent parameter.

> the authors show the collapse of the data for the whole range of system sizes? This is particularly important as the critical behavior with $\omega = 2$ is clearly not valid for small system sizes, such that one could expect to observe significant deviations. My final question is the limited range of W values shown in Fig. 8. In particular, the authors use this scaling behavior to recover the behavior of the boundary of the ergodic region $W^T(L)$, see Eq. (12). Therefore, their scaling hypothesis Eq. (11) should be valid up to the ergodic regime, i.e. for small values of W far from the transition point W_C. Could the authors show this scaling behavior in this regime?

The Referee is correct that our finite size scaling procedure leads to a systematic deviation of the results at $\overline{L} \leq 9 - 11$ (depending on the type of the random graph) and that this can be traced back to the behavior of $\overline{r} - \overline{r}_P$ in the vicinity of the critical point. However, the obtained data Fig. 8 are well collapsed for sufficiently large system sizes. As we discuss below, we do find it useful to include further sub-leading terms that would remedy this situation, as it is the behavior at the largest system sizes which is relevant in the thermodynamic limit.

As requested, in Fig. 4, we show our collapses for RRG with $D = 3$ in a wider range of disorder strengths. There are certain deviations from the scaling as $\overline{r}$ approaches the GOE value, which, however, become less significant as the system sizes increase (note that the $1/L$ behavior of $W^T_{\overline{r}}(L)$ is also observed only for few largest system sizes available to us).

> The authors state that they have used our scaling approach to analyse their data for RRG $D = 3$ and $D = 4$ and find critical exponents $\nu = 0.64$ and 0.67, and that they find deviations from our scaling for data with $\overline{r} \geq 0.4$ which is quite small and could indicate that our scaling behavior Eq. (5) would have for these models a very limited range of validity. I am surprised by these observations because I found I am able to fit accurately the data of the authors for these models with our assumption Eq. (5), using the critical

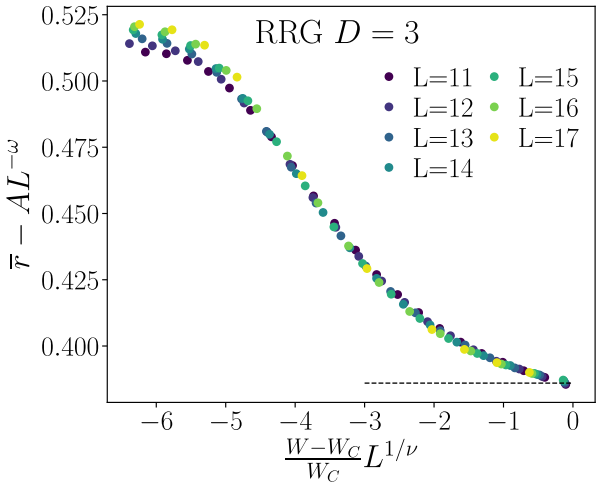

Figure 4: Data collapse as in Fig. 8(a) of the manuscript but in wider range of disorder strengths.

> disorder determined by the authors and the critical exponent taken as $\nu = 1/2$. More precisely, I fit the data with
>
> $$\overline{r} - \overline{r}_P = L^{-\omega} F(L^{1/\nu} w) \tag{5}$$
>
> equivalent to (5), with $w = (W - W_C) + A_2(W - W_C)^2 + A_3(W - W_C)^3$ and $F(X) = \sum_{k=0}^{5} B_k X_k$. In this analysis, the fitting parameters are the A_ks, B_ks and ω, whereas W_C and $\nu = 1/2$ are fixed. All curves for different W, in a range that I indicate for each model, are fitted simultaneously.

In the manuscript, we have written: "*the authors of [81] take a different approach than ours to the scaling. In particular, they study* $(\overline{r}(W, L) - \overline{r}_P)/(\overline{r}(W_C, L) - \overline{r}_P)$ *where* W_C *is the critical disorder strength, and find that an exponent* $\nu_\perp \simeq 1/2$ *dominates the scaling behavior close to the critical point. We have analyzed our data using their procedure (we thank Gabriel Lemarié for discussions regarding this) and we found that a similar phenomenology could be adopted to scale data for RRG with* $D = 3$ *(resp.* $D = 4$*) but with a different exponent* $\nu'_\perp = 0.64$ *(resp.* $\nu'_\perp = 0.67$*), instead of* $1/2$ *as dictated by data for SWN [81]*". It is clearly written that we adopted there the procedure of Ref. [81]. In that work the scaling (5) is used with $w = (W - W_C)$, see Fig. 3 of Ref.[81]. The resulting collapses for Anderson model on RRG are shown in Fig. 5, and yield the critical exponents 0.64 and 0.67, precisely the ones which are listed in our manuscript. Furthermore, if one enforces the exponent ν to be equal to 0.5, the data collapses are very poor already at data points corresponding to $\overline{r} \approx 0.4$, as written by us in the manuscript. Therefore, we believe that our claims in the manuscript are presented clearly.

Importantly, however, the story is much different when one uses $w = (W - W_C) + A_2(W - W_C)^2 + A_3(W - W_C)^3$ while fixing $\nu = 0.5$ as proposed by the Referee in his report. When this is done, the role played by ν is effectively assumed by the parameters A_2 and A_3 – this allows to obtain good data collapses by keeping $\nu = 0.5$. Below, we present further analysis of this point.

> This figure 3 shows quantitatively that the data of the authors close to the transition are also compatible with a critical exponent $\nu = 1/2$. I think the authors should compare the

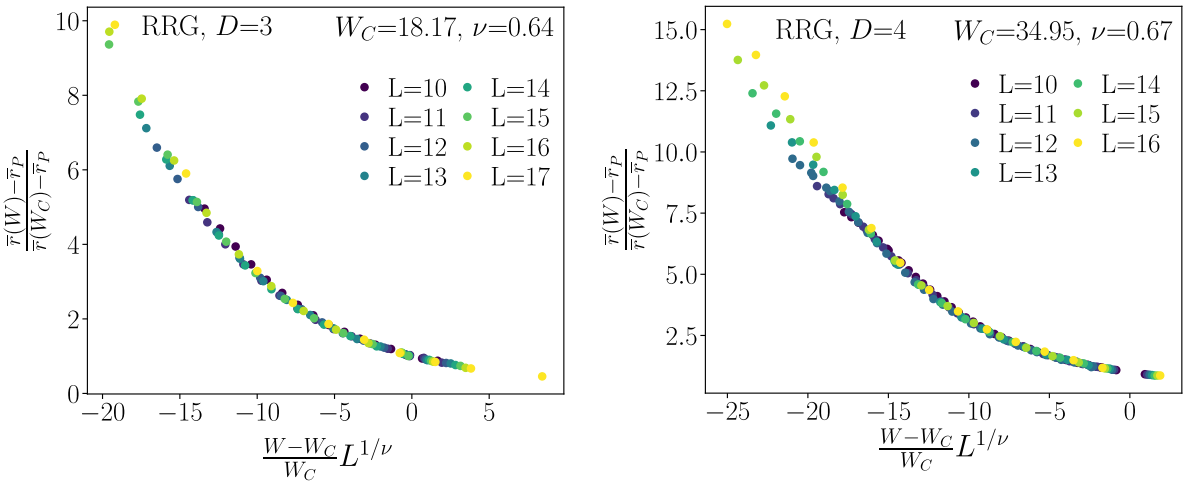

Figure 5: Collapses of $\frac{\overline{r}(W)-\overline{r}_P}{\overline{r}(W)-\overline{r}_P}$ as a function of $\frac{W-W_C}{W_C}L^{1/\nu}$ as proposed in [Phys. Rev. Research 2, 012020(R) (2020)] for RRG with $D = 3$ (the left panel) and $D = 4$ (the right panel). We fix the value of critical disorder strength W_C and treat ν as a fitting parameter.

> χ^2 they obtain from their fit with the χ^2 I have indicated, taking into account all system sizes in the range of W considered. After all, the scaling considered here is L/χ and one should allow for L to vary in the largest range to have a significant determination of the relevant scaling function and critical exponent.

In Fig. 6, we present a comparison of our scaling procedure with the finite size analysis proposed by the Referee. In the former case, we follow the analysis performed in our manuscript and show the values of χ^2, as defined by the Referee in his report. In order to make the former scaling more constrained, we use a second order polynomial $w = (W - W_C)/W_C + A_2(W - W_C)^2/W_C^2$. In our scaling procedure, Fig. 6(a),(c), we assume that $\omega = 2$, $\nu = 1$ and fit the only single parameter A, keeping $\nu = 1$ and $\omega = 2$. In contrast, there are two fitting parameters in the procedure suggested by the Referee, ω and A_2, while ν is kept as $1/2$. The values of χ^2 are comparable in all the considered cases showing similar quality of the collapses (importantly, for the same range of system sizes, i.e. considering only $L \geq 11$ we obtain $\chi^2 = 1.62e-05$ for RRG with $D = 3$ and $1.06e-05$ for RRG with $D = 4$ using the scaling proposed by the Referee).

There are two points that we would like to emphasize here. Firstly, the scaling procedure proposed by the Referee works better in a larger interval of system sizes. In contrast, our scaling procedure leads to systematic deviations when data for $L \leq 10$ are included. This is already apparent from the behavior of $\overline{r} - \overline{r}_P$ shown in Fig. 2. While we could remedy this by including a sub-leading term in our analysis (leading to two parameter scaling, as in the procedure proposed by the Referee), we opt not to do that as the data at system sizes $L \leq 10$ do not follow the same trends as data at larger L available to us. Secondly, the values of the term A_2 in the analysis proposed by the Referee are of the order or larger than unity. If the A_2 term was dominating, we would get $w \approx A_2(W-W_C)^2/W_C^2$ which means that the horizontal axis variable becomes $(W - W_C)^2/W_C^2 L^{1/\nu}$ which is equivalent to a collapse in terms of $(W - W_C)/W_C L^{1/(2\nu)}$. Therefore, in the limit of large A_2, $\nu = 1/2$ assumed by the Referee is the same as $\nu = 1$ assumed in our scaling assumption. Clearly, the values of A_2

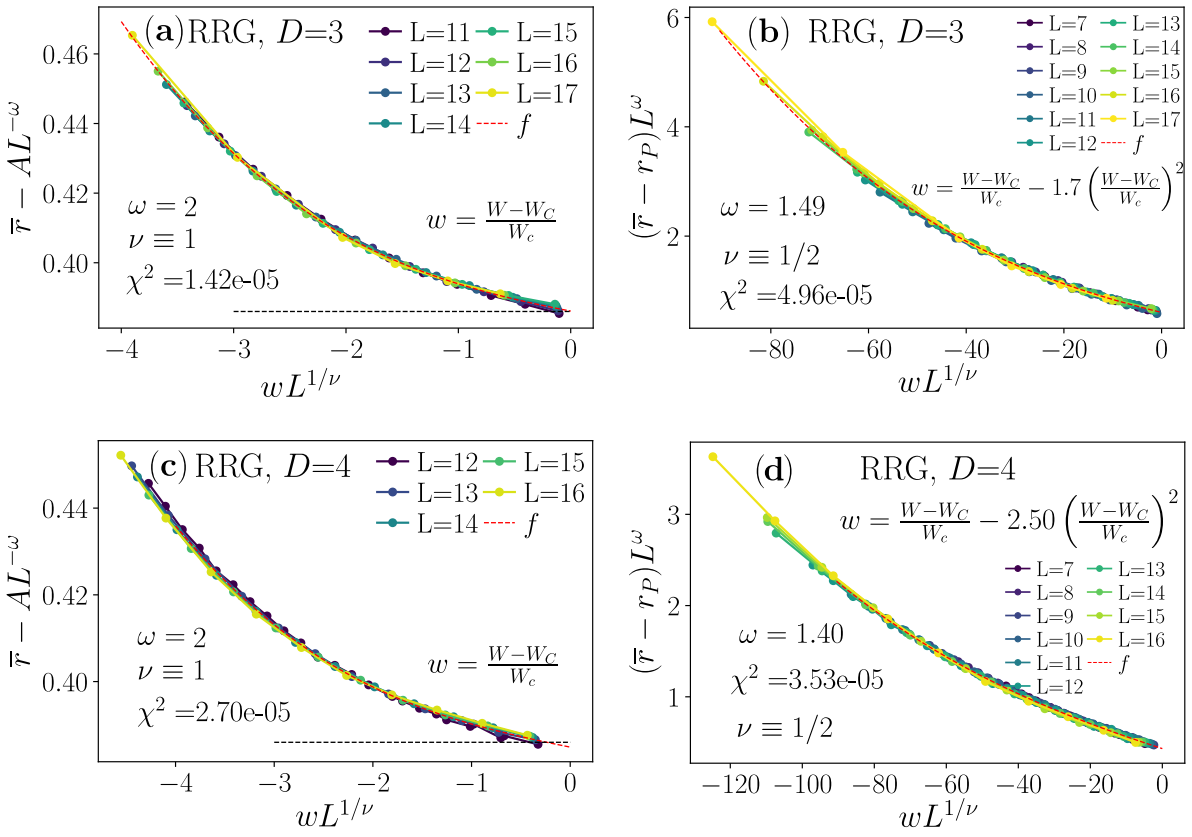

Figure 6: Comparison of the scaling analysis proposed in our manuscript, (**a**) and (**c**), with finite size scaling (5) proposed by the Referee, (**b**) and (**d**). The values of the obtained and assumed parameters are shown in the plot.

obtained by us are not orders of magnitude larger than unity, and both the linear and the quadratic term in w play a role. The combined effect of the two terms, is similar to assuming taking only the linear term $w = (W - W_C)/W_C$ and obtaining $\nu \approx 0.65$, as we did in Fig. 5.

For those reasons, we do not believe that the procedure proposed by the Referee is sufficient to clearly demonstrate that $\nu = 1/2$. To the contrary, both considered scaling procedures work similarly well in the relevant regime of large system sizes.

> I think the authors' data are precise enough to determine quantitatively which of the two scenarios, mainly $\nu = 1/2$ or $\nu = 1$ and $\omega = 2$ is more likely. I therefore invite the authors to make this quantitative comparison.

The detailed analysis presented above indicates that the data for Anderson localization transition are well described by both the approaches, preventing us for unambiguously deciding which of the critical exponents $\nu = 1/2$ or $\nu = 1$ is valid. We believe that both approaches have their advantages and disadvantages and only numerical data for larger system sizes can decide which of them is correct. However, we must point out that the difference between our two results backs-up two completely different analytic understanding of the transition. In our case, $\nu = 1$ is the exponent of the transition coming from the localized region which is undoubtedly correct, from iterative calculations dating back to Abou-Chacra, Anderson, and Thouless (a line of research which one could call the Bethe lattice works since they write a recursion equation which does not take into account the presence of loops, an approximation which is most probably correct in the localized region). We are further advancing that $\nu = 1$ describes the transition also from the delocalized region, providing a good collapse for the data coming from that region as well, as long as one irrelevant scaling function is added. Dr. Lemarié's scaling $\nu = 1/2$ is supposed to work well to describe the transition also in the localized region, therefore contradicting the Bethe lattice works. We do not think that there is enough numerics to support such a bold claim.

To give a full account about the two approaches to the Anderson transition on random graphs, we have added an Appendix to our work. The Appendix is referenced in Sec. 4.3 and it contains the most important points raised in our response to the Report by Gabriel Lemarié. In Acknowledgments we thank the Referee for the suggestions of the alternative scaling forms. We hope that the Referee will find our response to be convincing and our manuscript to be suitable for publication in SciPost Physics.