Reply to Anonymous Report 3

> Overall, the paper is well structured, and relevant in the field of Anderson transitions in tree-like structures. This work is important in its own right, and too many references to MBL are unwarranted. In particular, given several works on finite size/time effects of the many-body problem, and further understanding of correlation in Fock space, I think the MBL part of the motivation can be omitted.

We thank the Referee for the thorough reading and the positive assessment of our work. We fully agree that the problem of Anderson transitions on tree-like structures is important and interesting on its own. However, we also strongly believe that the analogies between the Anderson transition on random graphs and the MBL transition are worth exploring and constitute another motivation to investigate the former problem. Notably, this view is also shared by the Report 2 of Dr. Gabriel Lemarié, which states that the Anderson problem on random graphs is "a subject that has attracted much attention recently due to its analogy with the MBL transition".

> I find it absurd to compare the finite size disorder strength to MBL (XXZ+random field) problem with the graph problem, which as we understand now is just an analogy and can not be compared directly. I think the authors should not do it as it gives a different impression to readers.
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> Furthermore, several recent studies also suggest that the previously understood critical disorder strength is no where close to the true MBL transition. Therefore, caution is needed for naive comparison.

We fully agree that the MBL transition and Anderson transitions are at most analogous and conclusions for one of the transitions cannot be directly applied for the other. We expressed this thought for instance in Sec. 2: *"The Anderson model on random graphs provides a good reference point with which to compare, for the above described results <about the MBL transition>. On one hand, the phenomenology of the crossover between delocalized and localized regimes on random graphs is similar to the ETH-MBL crossover in many-body systems, as discussed in the preceding Section. On the other hand, in contrast to the many-body case, the critical disorder strength for Anderson model on random graphs can be precisely calculated as we show in Sec. 5".*

To highlight further the pointed raised by the Referee, we added in Sec. 2: *"Importantly, despite the analogies, the ETH-MBL crossover and the Anderson transition on random graphs are vastly different phenomena. The former depends crucially on the interparticle interactions, whereas the latter is a single particle problem on a random graph with uncorrelated on-site potentials. Nevertheless, a careful analysis of finite size effects at the Anderson transition on random graphs which we perform in this work may provide useful intuitions for the ETH-MBL crossover."*

> This is also well understood that the shift in the crossing point of the $\overline{r}(W)$ to stronger disorder suggests that the data is not in the scaling regime. At least in the usual Anderson transition, this is important to go to the scaling regime to do the scaling analysis. This is a reason why the MBL transition (and also the RRG problem) is so difficult to access. I understand that the authors were careful to call it a crossover, and not transition, but then what is the meaning of scaling collapse in a crossover regime?

The fact that the crossing point $W^*_{\overline{r}}(L)$ is shifting with system size L implies that $\overline{r}(W, L)$ is not well captured with a scaling form

$$\overline{r}(W, L) = \overline{r}_P + f((W - W_C)L^{1/\nu}), \tag{1}$$

where $f(x)$ is a certain function. For that reason we introduce a generic (see for instance [K. Slevin, T. Ohtsuki, Phys. Rev. Lett. 82, 382 (1999)]) sub-leading correction to the scaling $L^{-\omega} f_1((W - W_C)L^{1/\nu})$, which leads us to the equation 10 in the manuscript. Such a scaling form together with our minimal assumptions on the functions $f(x)$ and $f_1(x)$ predict that the crossing point shifts with system size L according to $W^* \simeq W_C - C_1 L^{-\omega/2-1/\nu}$ (equation 15 in the manuscript). In the absence of the sub-leading term, i.e. in the limit $\omega \to \infty$, there is no shift of the crossing point, $W^*(L) = W_C$. If $\overline{r}(W, L)$ was available for sufficiently large system sizes, the crossing point would be nearly system size independent $W^*(L) \approx W_C$. This is a desirable situation, but it is not the case for the present day numerical results even for the simpler Anderson model on regular lattices in 4-6D, see Figs. 7, 8, 9 of [E. Tarquini et al., Phys. Rev. B 95, 094204 (2017)]. Nevertheless, one can still perform a finite-size scaling of the data characterized by the shift of the crossing point (which is very strongly pronounced for Anderson model on random graphs, as discussed in the manuscript) provided that the sub-leading corrections to the scaling are included. In this sense, the data analyzed in our work *are in the scaling regime*. The finite size scaling ansatz in our eq. (10) *does* allow for a good scaling collapse of the data. However, the scaling forms with sub-leading corrections lead to a less constrained finite size scaling procedure which is the source of difficulties and controversies in the analysis of Anderson transition on random graphs.

In our manuscript we adopt a terminology in which the *crossover* is a gradual change of system properties at finite L, whereas *transition* is a singular change in system properties that occurs in the thermodynamic limit. We believe that our finite size scaling analysis performed for numerical data at the delocalization/localization crossover for Anderson model on random graphs predicts the critical properties of Anderson transition on random graphs.

> Even then I appreciate the author's careful comparison with finite-dimensional Anderson transition in regular lattice and by doing that justification of using two different definitions of W^*_T, to me it seems the severe finite size effects that are present in RRG problem is almost absent.

We are not fully certain whether we understand this remark of the Referee correctly. Our data for $W^*_{\overline{r}}(L)$ and $W^T_{\overline{r}}(L)$ clearly demonstrates that the finite system size drifts at the delocalization/localization crossover in the Anderson model on random graphs are significant (and to some degree analogous to what is observed for the problem of MBL). For finite-dimensional Anderson transition on regular lattice such drifts are much weaker, but still present. The important difference, which we emphasize at the end of Sec. 4.3, is that the critical point is associated with multifractality of the wave-function in the finite-dimensional case, whereas for random graphs one expects that $\overline{r}(W_C) \to r_P$ as $L \to \infty$.

> Fig. 5(a,d) and 6: the $W^T(L)$ does not show $\sim L$ scaling ! The largest system sizes significantly deviate from it - the authors need to take that into consideration and not ignore it and convince readers that even then $\sim L$ is justified. The authors pointed out that this indicates delocalized volume does not grow indefinitely. I think this conclusion is based on taking $p_{\overline{r}}$ as some arbitrary finite number. Why can even this number depend on L? Could it be true that the bending is just an artifact of this number?

We fully agree with the Referee that $W^T(L)$ does not show $\sim L$ scaling at sufficiently large system sizes. This is very important point of our analysis – the linear behavior $W^T(L) \sim L$ at small system sizes changes to a much weaker system size dependence $W^T(L) \sim 1/L$ at large L. This is emphasized in the Figures in our manuscript as well as written in the text explicitly, e.g. *"The results for RRG are shown in Fig. 5. Both for $D = 3$ and for $D = 4$, we find a linear with system size scaling of $W^T_{\overline{r}}(L)$ for $L \in [6, 14]$, and a clear deviation from this linear scaling at $L \geq 15$."* (see Sec. 4.2); *"Our results, at largest system sizes we can reach in the numerical computations, indicate two different scaling behaviors: $W^T_{\overline{r}} \sim L^{-1}$ ..."* (see Sec. 4.3). The behavior $W^T(L) \sim 1/L$ heralds the localization transition which occurs at critical disorder strength W_C which is evaluated with the cavity method in Sec. 5 and depends on the type of random graph. We have verified that our conclusions about the behavior of $W^T(L)$ are independent of the value of $p_{\overline{r}}$ in the interval $[0.002, 0.06]$ (which is quite broad given the difference between r_{GOE} and r_P). This is illustrated by the points $W^T_{\overline{r}=0.47}$ shown in Fig. 5(b). The same conclusion holds for other types of random graphs, indicating that our results are robust with respect to choice of the value of $p_{\overline{r}}$. Finally, we would like to note that the system size dependence of $W^T(L)$ is analogous to certain models of MBL which exhibit a regime of a linear behavior $W^T(L) \sim L$ which is replaced by a weaker system size dependence at larger L.

> I come back to the same question once more - why even though the system sizes are not in the 'true' scaling limit, but would give correct scaling collapse as shown in Fig. 8. Request is to show the data with open symbols, and without the red line. I have the feeling that the data shows significant scattering, and this is not a true collapse but an approximate one. Moreover, it should be noted that the quality of the data is so good, that any scattering can not be taken as an artifact of less sampling!

As we have written above, our data are in the scaling regime provided that one considers a finite-size scaling with the relevant and one irrelevant variable. Below, we show a version of Fig. 8 with data with open symbols, without red lines and insets, for better visibility of data. We do not observe a systematic deviations in the reported collapses, hence we believe that the scaling ansatz we assumed describes the numerical data well.

> **Requested changes**
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> 1. To begin with my humble request to the authors to show the collapse data with open symbols. This would help the reader to understand the quality of the collapse.
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> 2. Slightly different motivation should be incorporated and MBL part can be shortened, and its reference through out the paper.

1. We have changed Fig. 8 in the manuscript denoting the collapsed data with open symbols and adjusting the width of the red lines. We hope that the data presented in that way are more readable.

2. As we have argued above, the analogies between the Anderson localization transition on random regular graphs and the MBL transition constitute for us an important motivation of our work. Moreover, one of the results of our work is that the finite size drifts of $W^T_{\overline{r}}$, $W^*_{\overline{r}}$ are to large extent similar to the drifts reported in the works on MBL transition. The investigations of the Anderson transition on random graphs allow us to demonstrate what finite size effects can be encountered in exact diagonalization studies of models in which the presence (and the position) of the localization transition is known. This, in our view constitutes an important reference point for

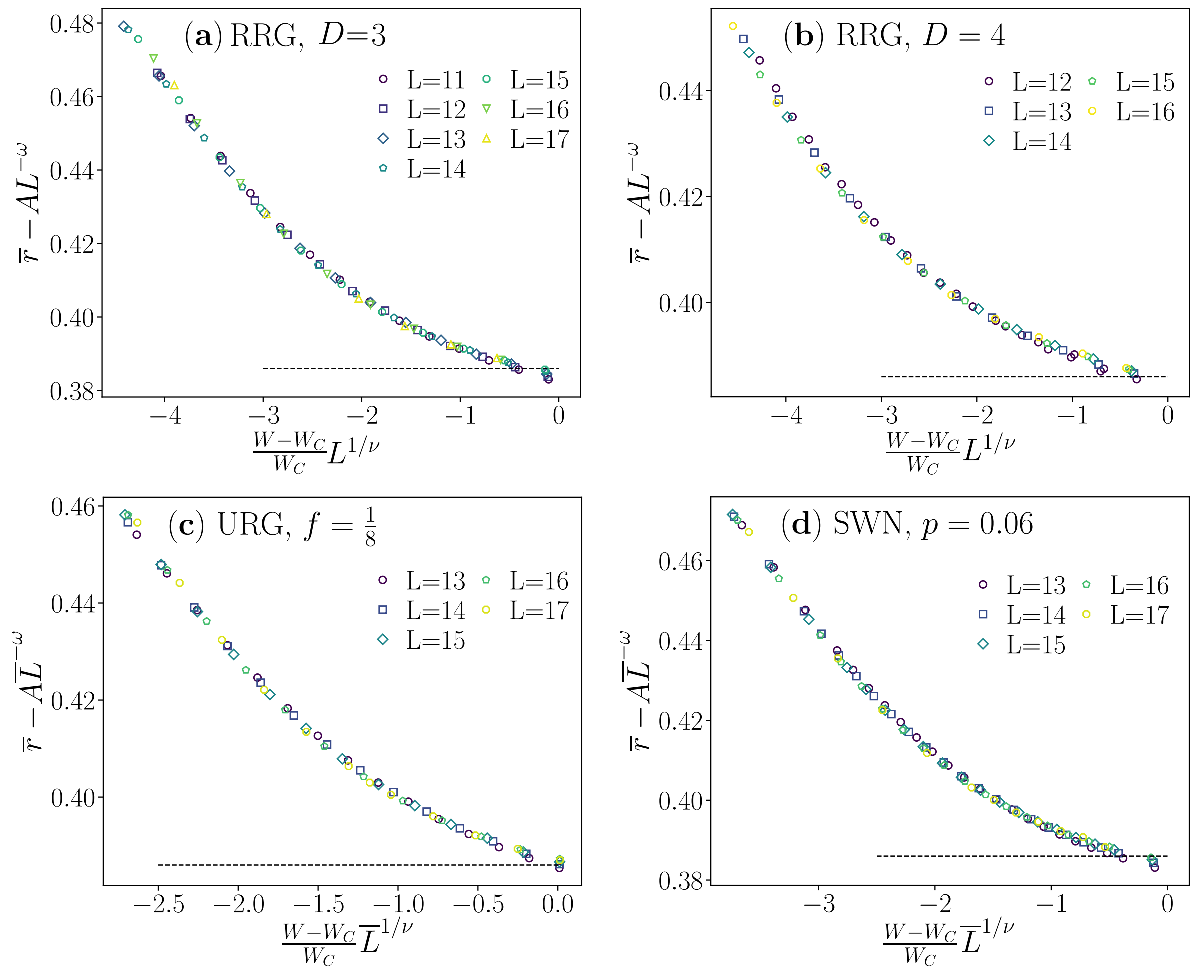

Figure 1: Finite size scaling of data at Anderson transition on RRG with $D = 3, 4$, panels ((**a**), (**b**), for URG with $f = \frac{1}{8}$, panel (**c**), and for SWN, panel (**d**). In all cases we set $\nu = 1$, $\omega = 2$ (see text) and the critical disorder strength W_C evaluated in thermodynamic limit; the *single free parameter* of the presented finite size collapses of the data is the coefficient A in the term $AL^{-\omega}$.

MBL investigations, in which the position (or even the presence) of the transition is not so well certified. For those reason, we believe that the references to the MBL literature are well justified in our paper and we decided not to change this aspect of our work.