We thank the referee for the detailed report and for raising some important issues.

We will clarify the notion of even characters through some calculations and mention the proposed changes we would incorporate in the manuscript at the end of this post.

**Cancellation of poles in even characters:**

We illustrate the cancellation of poles of the even characters in the example $\widehat{su(2)}_{-1/2}$ mentioned by the referee. Next, we show the stated cancellation for an arbitrary admissible level by writing the character as a $z$-series ($\omega = e^{2\pi i z}$).

We begin by the $\widehat{su(2)}_{-1/2}$ example. In our notation, the level $m$ parametrised as $m = p/u - 2$ implies $p = 3$, $u = 2$. The spins are $j(0,1) = -3/4$ and $j(1,1) = -1/4$. The character $\chi_{-3/4}$, labelled by its spin is,

$$\chi_{-3/4}(q,\omega) = q^{-1/12}\omega^{-3/4}\frac{\sum_{s\in\mathbb{Z}} q^{6s^2+s}\omega^{3s} - \sum_{s\in\mathbb{Z}} q^{6s^2+1-5s}\omega^{3s-1}}{\sum_{s\in\mathbb{Z}} q^{2s^2+s}\omega^{2s} - \sum_{s\in\mathbb{Z}} q^{2s^2-s}\omega^{2s-1}}\,.$$

After the polynomial division we have,

$$\begin{aligned}\chi_{-3/4}(q,\omega) &= -q^{-1/12}\omega^{1/4}\left(1+\omega+\omega^2+\cdots\right) - q^{11/12}\omega^{1/4}\left(1+2\omega+2\omega^2+\cdots\right)\\ &\quad + \mathcal{O}(q^{23/12})\,, &(1)\\ \chi_{-3/4}(q,|\omega|<1) &= -q^{-1/12}\frac{\omega^{1/4}}{1-\omega}\left(1+q(1+\omega)+\mathcal{O}(q^2)\right)\,. &(2)\end{aligned}$$

We used,

$$\frac{1}{1-\omega} = 1+\omega+\omega^2+\cdots\,,\qquad |\omega|<1\,. \tag{3}$$

Similarly, the other character with spin $j = -1/4$ has the expansion,

$$\begin{aligned}\chi_{-1/4}(q,\omega) &= -q^{-1/12}\omega^{3/4}\left(1+\omega+\omega^2+\cdots\right) - q^{11/12}\omega^{3/4}\left(\frac{1}{\omega}+2+2\omega+2\omega^2+\cdots\right)\\ &\quad + \mathcal{O}(q^{23/12})\,, &(4)\\ \chi_{-1/4}(q,|\omega|<1) &= -q^{-1/12}\frac{\omega^{3/4}}{1-\omega}\left(1+q\left(\frac{1}{\omega}+1\right)+\mathcal{O}(q^2)\right)\,. &(5)\end{aligned}$$

Indeed, the naive character subtraction, upon subtracting eqns (1) and (4) gives the infinite series, one with positive coefficients and the other with negative coefficients, but in the region $|\omega| < 1$, it is easy to see that the even characters obtained by subtraction of (2) and (5),

$$\begin{aligned}\chi^{+}_{-3/4}(q,\omega) &= \chi_{-3/4}(q,\omega) - \chi_{-1/4}(q,\omega)\,, &(6)\\ &= -q^{-1/12}\frac{\omega^{1/4}}{1-\omega}\left(1+q(1+\omega)+\mathcal{O}(q^2)\right) + q^{-1/12}\frac{\omega^{3/4}}{1-\omega}\left(1+q\left(\frac{1}{\omega}+1\right)+\mathcal{O}(q^2)\right)\,, &(7)\\ &= -q^{-1/12}\frac{\omega^{1/4}(1-\omega^{1/2})}{1-\omega}\left(1-q\left(\frac{1}{\sqrt{\omega}}+\sqrt{\omega}\right)+\mathcal{O}(q^2)\right)\,, &(8)\end{aligned}$$

have a valid $q$-expansion in the limit $\omega \to 1$,

$$\chi^{+}_{-3/4}(q,\omega\to 1) = -q^{-1/12}\frac{1}{2}\left(1-2q+\mathcal{O}(q^2)\right)\,. \tag{9}$$

The limit also gives the fractional coefficients of 1/2 in the $q$-series (in general, we get $1/u$ as seen in the expansion eqn (2.19) in the paper).

Note that the region of convergence of the two characters $\chi_{-3/4}(q,\omega)$ and $\chi_{-1/4}(q,\omega)$ used to define an even character is the same, $|q|<1$, with $|q|<|\omega|<1$.

The character expansion for $|q|<1$, and $|q|^{-1}>|\omega|>1$, is obtained from

$$\chi_{-1/4}(q,|\omega|<1) = -q^{-1/12}\frac{\omega^{3/4}}{1-\omega}\left(1+q\left(\frac{1}{\omega}+1\right)+\mathcal{O}(q^2)\right)\,, \tag{10}$$

by change of variables, $\omega'=1/\omega$,

$$\chi_{-1/4}(q,|\omega'|>1) = -q^{-1/12}\frac{\omega'^{1/4}}{\omega'-1}\left(1+q\left(\omega'+1\right)+\mathcal{O}(q^2)\right)\,, \tag{11}$$

$$= -\chi_{-3/4}(q,|\omega'|<1)\,. \tag{12}$$

Thus, the character combination,

$$\chi_{-1/4}(q,|\omega|>1)+\chi_{-3/4}(q,|\omega|<1)=0 \tag{13}$$

This matches with the observations in [13]. It is also stated in [20] as,

$$\chi_{j(n,k)}(q,z) = -\chi_{j(n,k)}(q,-z)\,,\quad k\neq 0. \tag{14}$$

Now, we move on to demonstrate the cancellation of the pole at $z=0$ starting with eqn (2.16) in the paper. In the following expressions we will write characters with the numbers $b_\pm\equiv b_\pm(n,k)$ unless explicitly written.

$$\chi_{j(n,k)}(q,z) = \frac{1}{2\pi i z\eta^3(q)}\left(q^{b_+^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2+sb_+} - q^{b_-^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2+sb_-}\right)+\mathcal{O}(z)$$
$$+\frac{1}{u\eta^3(q)}\left(q^{b_+^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2+sb_+}(as+b_+/2) - q^{b_-^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2+sb_-}(as+b_-/2)\right). \tag{15}$$

Similarly, we can write the character expansion of $\chi_{j(\bar{n},\bar{k})}(q,z)$, using $\bar{n}+1=p-(n+1)$ and $\bar{k}=u-k$,

$$b_+(\bar{n},\bar{k}) = u(\bar{n}+1)-\bar{k}p = -b_+(n,k) = -b_+\,,$$
$$b_-(\bar{n},\bar{k}) = -u(\bar{n}+1)-\bar{k}p = -2a-b_-(n,k) = -2a-b_-\,.$$

$$\chi_{j(\bar{n},\bar{k})}(q,z) = \frac{1}{2\pi i z\eta^3(q)}\left(q^{b_+^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2-sb_+} - q^{(b_-+2a)^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2-s(b_-+2a)}\right)+\mathcal{O}(z)$$
$$+\frac{1}{u\eta^3(q)}\left(q^{b_+^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2-sb_+}(as-b_+/2) - q^{(b_-+2a)^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2-s(b_-+2a)}(as-(b_-+2a)/2)\right). \tag{16}$$

We focus on the part which has a pole at $z=0$,while the remaining piece gives the equation (2.26). Firstly, in the summation $s\to -s$, since $s\in\mathbb{Z}$ gives,

$$\chi_{j(\bar{n},\bar{k})}(q,z) = \frac{1}{2\pi i z\eta^3(q)}\left(q^{b_+^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2+sb_+} - q^{(b_-+2a)^2/4a}\sum_{s\in\mathbb{Z}}q^{as^2+s(b_-+2a)}\right). \tag{17}$$

Then, in the second sum over $s$, we change $s \to s-1$, which gives,

$$\begin{aligned}
\text{Pole}\left(\chi_{j(\bar{n},\bar{k})}(q,z)\right) &= \frac{1}{2\pi i z \eta^3(q)}\left(q^{b_+^2/4a}\sum_{s\in\mathbb{Z}} q^{as^2+sb_+} - q^{b_-^2/4a+a+b_-}\sum_{s\in\mathbb{Z}} q^{as^2+a-2as+sb_-+2as-2a-b_-}\right), \\
&= \frac{1}{2\pi i z \eta^3(q)}\left(q^{b_+^2/4a}\sum_{s\in\mathbb{Z}} q^{as^2+sb_+} - q^{b_-^2/4a}\sum_{s\in\mathbb{Z}} q^{as^2+sb_-}\right). \qquad (18)
\end{aligned}$$

This cancels with the pole piece in eqn (15) upon taking the difference in the even character,

$$\chi^+(q,\omega) = \chi_{j(n,k)}(q,\omega) - \chi_{j(\bar{n},\bar{k})}(q,\omega)\,, \qquad (19)$$

with the radius of convergence of the even character defined by $|q|<1$, and $|q|<|\omega|<|q|^{-1}$.

**Proposed corrections:**

1. We will correct the radius of convergence of the even character in the conclusions and add to the definition of the even characters the radius of convergence $|q|<1$, $|q|<|\omega|<|q|^{-1}$.

2. To make the definition of the even character more transparent, we will add two illustrative examples of $\widehat{su(2)}_{-1/2}$ and $\widehat{su(2)}_{-4/3}$.