Referee reply — May 16, 2023

Daniel R. Mayerson

I would like to thank the two referees for their thoughtful comments and suggestions.

I have adapted this new version to discuss at length the well-definedness of ω^I, which the referees were entirely justified in pointing out as a glaring omission of the first version that requires resolution. Specifically, I now provide the necessary "gauge-fixing" conditions (that I had implicitly assumed before, i.e. eq. (66)). See immediately below for more details on what has been added in this respect. Below I also discuss each one of the two referees' specific comments and indicate what has accordingly been changed in the paper. Note that equation and citation references here refer to the numbering in the new version.

Well-definedness of the improvement form ω^I

As the referees point out, there is a "gauge" ambiguity $\boldsymbol{\omega}^I \to \boldsymbol{\omega}^I + dA^{(0)}$ for a seemingly arbitrary scalar $A^{(0)}$, which would affect the multipoles since the twist scalar changes as $\boldsymbol{\omega} \to \boldsymbol{\omega} + 2A^{(0)}$ under this "gauge" transformation. Section 3 has been reorganized, and in particular Section 3.3 is new and deals with this ambiguity.

The ambiguity must be resolved by a "gauge" choice which essentially demands that $\boldsymbol{\omega}^I$ "does not contain multipoles". This can be made precise in a coordinate-invariant way in the linearized theory (see Section 3.3.1) by defining a "multipole structure" for $\boldsymbol{\omega}^I$ (similarly to how Geroch originally defined multipoles (in flat space) for harmonic functions in a coordinate-invariant way), and then demanding that these multipoles all vanish. At non-linear order (see Section 3.3.2), this "gauge" condition can be written as eq. (66), which is the ACMC-coordinate version of demanding that $\boldsymbol{\omega}^I$ "does not contain multipoles".

The discussion at linear order is further illustrated with the improvement form for Einstein-Maxwell theory, essentially showing that the stated form of $\boldsymbol{\omega}^I$ is indeed the unique one for this theory.

Incidentally, two additional minor discussions have been added to the paper to further support the new Section 3.3. First of all, it now contains a brief discussion of the conformal structure of the Einstein equations as expressed in terms of the scalars $\Phi_{M,J}$ (as derived by Hansen — eqs. (17)-(19)) and their extension to the non-vacuum case (eqs. (43)-(52)). Second, the interpretation of the Geroch-Hansen formalism in terms of the (timelike) Kaluza-Klein reduced metric is given — and in particular gives the origin of the "equation of motion" (50) that $\boldsymbol{\omega}^I$ must satisfy.

Specific responses to Referee 1

1. *(On ambiguity of ω_i^I and the resulting ω.)* This point has been addressed above. (Note also that bold font face is now used in the paper for forms, as suggested.)

2. *(On the applicability when matter is not isolated.)* Indeed, as the referee points out, if the matter is isolated to a certain region of spacetime, then the Thorne formalism simply holds without alterations; Thorne indeed only needs the asymptotic structure, i.e. far away from the matter. (It is perhaps less obvious a priori how the Geroch-Hansen formalism will hold, since there it is assumed that the potentials $\Phi_{M,J}$ can be globally well-defined and conformally transformed to $\tilde{\Phi}_{M,J}$; however, one can presumably relax the Geroch-Hansen arguments to simply focus on (and conformally compactify) the region of spacetime where there is no matter and the formalism can

be used to conformally compactify/transform. After all, only the analytic structure of $\tilde{\Phi}_{M,J}$ near the point at infinity Λ is needed.)

However, as the referee points out, it is indeed a priori not necessary that the ACMC expansion (20) will continue to hold in the presence of non-isolated matter (i.e. extending to infinity). The existence of the ACMC expansion (20) is indeed an *assumption* when non-isolated matter is present. This is mentioned at the beginning of Section 3. Providing some evidence for the "naturalness" of this assumption — even in the presence of matter — is precisely the goal of the discussion in Section 3.4. (See, for example, the third paragraph, "Beyond vacuum spacetimes, the smoothness of the metric at infinity is not guaranteed, nor is the existence of a suitable ACMC coordinate system.")

3. *(On the definition of asymptotic flatness.)* I indeed implicitly assumed the definition of asymptotic flatness as used by Geroch and Hansen (which is necessary in order for their formalism to apply). A paragraph has been added in Section 2.1 to explicitly state this definition of asymptotic flatness and that I assume this definition for the notion of asymptotic flatness. It¸ is now also explicitly re-stated at the beginning of Section 3 (second paragraph) that this definition of asymptotic flatness is also used for non-localized matter. (Of course, the condition of asymptotic flatness implicitly demands specific asymptotic fall-offs for the energy momentum tensor through the Einstein equations.)

4. *(On the discussion of [11].)* I agree that it is of course not true that the two families of gravitational multipoles completely determine a metric in non-vacuum; a simple illustration are the identical multipoles of the Kerr and Kerr-Newman solutions. It is important to note that the existence of an ACMC expansion (20) certainly does not imply that the metric can be reconstructed from these the gravitational multipoles (again, Kerr-Newman, which can be brought to ACMC form, is a simple example). I have added a footnote (footnote 2 on p2) to emphasize this point (although I admit I am not entirely certain this is the place that the referee was referring to when stating the necessity of rephrasing how [11] is cited).

5. *(On the first sentence of the paper.)* The first sentence has been changed from *"Multipoles are usually thought of as coefficients in an asymptotic radial expansion of a field."* to *"Multipole moments of a field encode the angular structure of the field as determined by its sources; successive multipole moments can typically be read off from the angular dependence of terms in an asymptotic radial expansion."*.

6. *(On the radial dependence of $\mathcal{C}$.)* The function $\mathcal{C}$ has the same $1/r$ fall-off as the other functions; not mentioning this was an oversight on my part. The fall-off of $\mathcal{C}$ has been added to (75).

7-9. *(Typos.)* The three indicated typos have been addressed.

Specific responses to Referee 2

1. *(On the well-definedness of ω^I_μ, and also in particular in the case of Einstein-Maxwell.)* The well-definedness of ω^I_μ in the general case was addressed above.

   As for the Einstein-Maxwell electrostatic potentials $\rho, \tilde{\rho}$, specifically, note that these *are* in fact unique, as long as $\mathcal{L}_\xi \boldsymbol{A} = 0$, i.e. the gauge potential $\boldsymbol{A}$ is stationary. As already shown in Appendix B.1, it is always possible to find a gauge such that

$\mathcal{L}_\xi \boldsymbol{A} = 0$ as long as the field strength is stationary, $\mathcal{L}_\xi \boldsymbol{F} = 0$. I have added Appendix B.2 to show explicitly that $\rho, \tilde{\rho}$ are then unique under this stationarity condition. A note referring to this Appendix has also been added in Section 3.1.1, where the Einstein-Maxwell improvement form ω^I_μ is discussed. The "gauge fixing" conditions of Section 3.3 are also explicitly checked on the Einstein-Maxwell theory, confirming that the given improvement vector is indeed the unique correct one.

2. *(On relaxing the conditions of asymptotic flatness.)* There have indeed been attempts to generalize the concept of multipoles to non-asymptotically flat spacetimes, e.g. to dS spacetimes. However, Ref. [11] does not discuss dS spacetimes, but 2105.09971 does. This reference has been added and referenced in Section 1.1, i.e. [31].

3. *(On relaxing the conditions of stationarity, and 1008.1278)* The Noether charge formalism of Ref. [11] indeed holds for non-stationary spacetimes, and it would be interesting to expand the current work to include a comparison with the Noether charge formalism. However, since the current work focuses mostly on the Geroch-Hansen formalism — which is inherently (only) defined for stationary metrics —, I leave this to possible future work. A sentence has been added to this effect in Section 1.1.

   I have added a reference to 1008.1278, i.e. [30], in Section 1.1. This work indeed deals with attempting to circumvent the non-zero matter fields at infinity for the specific case of (linearized) bumpy black holes. However, as now mentioned in Section 1.1, note that these metrics cannot be brought to ACMC-N form for arbitrary N and correspondingly applying the Geroch-Hansen formalism is also not possible for all multipoles (as was noticed in 1008.1278).

4. *(On providing an example of the generalized formalism.)* Unfortunately, the Kerr-Vaidya spacetime is not stationary so does not fall in the class of metrics considered here. The bumpy black holes of 1008.1278 also do not fall under this formalism as mentioned above.

   As for other examples, whereas there is no specifically "new" example of multipole moments calculated in this paper, what this paper provides is a justification and explanation for various earlier applications of multipoles where $T_{\mu\nu} \neq 0$, for example in higher derivative theories (Ref. [22]) or theories with non-zero massless fields at infinity (eg. Refs. [15-19]). This is mentioned towards the end of the Introduction.[1]

[1] Note further that the (Jordan-)Brans-Dicke solutions mentioned in footnote 13 could be viewed as "new" applications of the ACMC formalism, but here trivially $\boldsymbol{W}^{(2)} = \boldsymbol{\omega}^I = 0$ so the improvement form does not really feature.