Anonymous Report 2 on 2023-2-2 *(Invited Report)*

Report

The bulk-boundary correspondence (BBC) is a universal feature in the Hermitian system and is essential in understanding and classifying the topological phases. In this paper, the authors generalized this idea to non-hermitian (NH) systems and investigated BBC in NH topological phases. In particular, they focused on a one dimensional energy band with a point gap and showed that there exists a one-to-one correspondence between the bulk topological invariant and the localized modes on the boundary. This paper is well written and includes interesting results. Therefore I recommend it to be published on SciPost after the authors address the following questions:

We thank the Referee for acknowledging the relevance of our results and for recommending our work for publication.

1 I find the derivation for the BBC is rigorous. However, I do have a practically question about these localized singular vectors. The NH Hamiltonian effectively appears in the open quantum system coupled with an environment. The authors used the singular value spectrum to characterize the topological features of this effective Hamiltonian. Can the authors elaborate on how to measure these singular value spectrum experimentally?

We thank the Referee for regarding our derivation of the BBC as rigorous. Our proof of the BBC relates in a one-to-one fashion the integer value of the winding number Eq. (19) under periodic boundary conditions (PBC) to the number of (exponentially) localized singular vectors present under open boundary conditions (OBC). These localized singular vectors are the physically relevant edge modes and we refer to them as "zero singular modes" (ZSMs), since they are associated with singular values which are exponentially small in system size. This feature allows us to draw a direct connection to the observability of the ZSMs, for the following reason. As we show in Eq. (42), if we are in a topologically non-trivial regime, the ZSMs dominate the response of the system [the susceptibility is closely connected to the inverse of the NH Hamiltonian, see Eq. (41)]. As a consequence of that, the response to a weak coherent probe becomes exponentially large in system size. From an experimental point of view, one can easily measure the output light at any given site (downstream with respect to the input probe) in a simple transmission experiment: detecting amplification in the transmitted light which features (i) directionality (the left singular vectors select the input sites and the right singular vectors the output sites with the largest gain) and (ii) exponential gain is a unique signature of the ZSMs and can be used to measure them. Indeed, it is possible to directly extract the ZSMs from scattering experiments. When probing at the first site, the output light along the chain is proportional to the superposition of the (right) zero singular modes and the gain is proportional to the inverse of the singular value. E.g., in the case L=1, the vector of output fields is proportional to the single, right ZSM which can therefore be measured directly. This feature can be appreciated in Fig. 6, where we show how two localized singular vectors are associated with two distinct channels of directional amplification, both featuring exponential gain. The amplified response observed at the last and second-to-last site of the array can be seen as a direct measurement of the zero singular modes and in turn, via the reinstated BBC, of the topological invariant. In fact, we regard the direct link between a measurable quantity and NH topology one of the most relevant contributions of our work.

In Sec. X, we added the following sentence at the end of the paragraph after Eq. (42) "In a topologically non-trivial phase, thanks to Eq. (42), the response to a probe field is proportional to the right ZSM $\alpha(\omega) = -\sqrt\gamma\chi(\omega)\alpha_\mathrm{in}(\omega) \propto v_0$, which provides a clear physical interpretation of the ZSM." and we also modified the following sentence in the second paragraph after Eq. (42) to read "NH topological amplification entails that the ZSMs are directly measurable in a simple transmission experiment and the topological winding number (19) can be extracted by counting the number and direction of amplified edge modes."

2 If we consider a hermitian topological phases subject to dissipation, we then can obtain a NH Hamiltonian. Do we expect to see such a BBC in this NH Hamiltonian?

We thank the Referee for their insightful comment. The non-Hermitian BBC that we derive applies to one-dimensional non-Hermitian tight-binding Hamiltonians featuring a single band that forms a closed curve in the complex plane (point gap). The topological properties of this class of systems can only be described in terms of non-Hermitian topology, i.e. there is no limit in which the system reduces to a Hermitian model that supports a topologically non-trivial phase. We stress that this class of models is highly relevant. Indeed, point gapped Hamiltonians have proven to be the most challenging from the point of view of topology, given that all of them exhibit the non-Hermitian skin effect and so seemingly undermine the BBC. The main contribution of our work is to show how to reinstate the BBC for this class of non-hermitian Hamiltonians.

Having said that, one of the simplest Hermitian models featuring non-trivial Hermitian topology to which one can add decay is the SSH model. Such a non-Hermitian SSH model has been studied by several authors, e.g. by Kunst et al., [Phys. Rev. Lett. 121, 026808 (2018)] and [E. Edvardsson et al., Phys. Rev. Research **2**, 043046 (2020)], who employed the formalism of bi-orthogonal quantum mechanics to show that the conventional BBC breaks down but a BBC be restored considering the “bi-orthogonal polarisation”.

3 The symmetry plays a key role in classifying the topological band structure in the Hermitian system and protects the zero modes on the boundary. Do we observe similar phenomena in the NH band theory?

Connected to our response to the previous point, the notion of topology we refer to in the paper applies to one-dimensional NH Hamiltonians with a point gap. We reiterate that point-gapped Hamiltonians constitute the most relevant class of NH topological models. Despite featuring a single complex band and no additional symmetry (other than translational invariance), they can still be topologically nontrivial, according to NH topology. Therefore, contrary to Hermitian topology, NH topological phases can occur even in the absence of symmetries. We stress this important point several times in the manuscript, e.g. in the abstract where we write: “Here, we restore the bulk-boundary correspondence for the most paradigmatic class of NH Hamiltonians, namely those with one complex band and without symmetries”.

Going beyond point-gapped spectra and to include the role of symmetry protection in NH bands would certainly be an interesting direction to investigate. We write explicitly in Sec XIII, Conclusion and outlook: “Multi-band models can endow the NH Bloch Hamiltonian with symmetries, for which a classification in terms of 38 symmetry classes was recently proposed [Gong et al. PRX 2018, Kawabata PRX 2019]. The impact of these symmetries on the system’s transport properties has been unexplored so far, which would be an ideal task for our framework”. This however goes beyond the current manuscript and will be addressed in the future.

We thank the Referee for their careful review and hope they can support publication of our manuscript.