In the following, we mention the comment by the referee in italics, followed by our response and the proposed changes to address the corresponding points.

1. *The end of the second paragraph suggests that CFTs with a finite number of primaries are rational. This is false and the original counterexample is the triplet model, see hep-th/9604026.*

   Thanks for pointing this out. We would change the end of the second paragraph to the following:

   "Contrast this situation to the theories at positive integer levels. At positive integer levels, the Wess-Zumino-Witten (WZW) models have a finite number of irreducible unitary highest-weight representations of the current algebra that close under fusion. Each such representation has a finite energy eigenspace. Thus, the positive integer level WZW models are *rational* in the sense described in Appendix A.3 of [1]. Footnote: Note that a finite number of primaries in the CFT does not ensure its rationality. See for example [2]."

2. *The word "representation" appears to be used frequently to mean "irreducible highest-weight representation" or something similar (but perhaps not always). Perhaps it could be clarified that this is the case unless otherwise noted?*

   We would add a footnote in the introduction to the same effect before eqn (1.1) in the original manuscript.

3. *Footnote 3 claims that "admissible" will be defined in the next section. But, I didn't see it defined there (or anywhere else). Is it the Kac-Wakimoto definition from their 1988 paper, so only applicable to irreducible highest-weight representations? Or does it mean any representation of the vertex operator algebra underlying the CFT under consideration? Or something else?*

   We define the admissible (unflavoured) characters which satisfy the properties mentioned at the beginning of section 2 in the manuscript. We would rephrase the paragraph below eqn (2.1) to state this clearly.

   " The coefficients $a_i^{(j)}$ need to satisfy certain admissibility conditions on the characters. We impose the following conditions on the unflavoured characters, which are necessarily satisfied by the characters of an RCFT. We define *admissible characters* as the unflavoured characters which satisfy the following conditions."

4. *The term "big category of admissible modules" after (1.1) is perplexing, especially as reference is made to [17] where the term is never used. Please be more precise here.*

   "Big category" was sloppy on our part. We will replace it with extended category, which we will explain. We will also replace the reference [17] to [11,13] where this is shown.

   "The highest weight characters can be chosen to be the linearly independent characters amongst the set of characters corresponding to the extended category of admissible modules, which includes the relaxed category and the indecomposables [3, 4]."

5. *There appears to be a systematic misunderstanding of the term "integrable" in the paper. As far as I can tell, it is universally understood (in this context) to mean that a representation decomposes as a direct sum of finite dimensional $su(2)$ representations, for any choice of $su(2)$ subalgebra. As such, $SU(2)_k$ has no (non-zero) integrable highest weight representations unless k is a non-negative integer. However, the author mentions them in many places for fractional levels. Perhaps they are thinking of representations whose characters converge at $z = 0$, ie. finite-dimensionality for one special choice of $su(2)$ subalgebra?*

Thanks for the definition. Indeed, we misunderstood. We will remove the word integrable from several places where a wrong meaning is evident. In the remaining places, the integrable representations only apply to the irreducible highest weight representations of $\widehat{su(2)}_k$, $k \in \mathbb{Z}_{>0}$. We will highlight these changes in the revised manuscript.

6. *In the second paragraph of Sec. 1.1, what does "admissibility of the even characters" mean? These objects don't correspond to representations in general. Do you mean both the representations whose characters are being subtracted are admissible?*

   The admissibility of even unflavoured characters (after multiplying by a suitable integer such that the resulting q-series has integer coefficients) borrows the definition of admissible characters we used in section 2 of the manuscript (also point 3 in this post). We would clarify by adding a statement in the second paragraph of section 1.1.

   "The even characters are admissible if they satisfy the admissibility conditions defined in section 2."

7. *After (2.4), the RCFT discussion suggests that we add the constraint of positivity of fusion coefficients if the S-matrix is unitary. But, the S-matrix of a RCFT is always unitary...*

   The S-matrix of eqn (2.4) (in the manuscript) need not always be unitary. It is unitary if the matrix $M = \mathbb{I}$ (of eqn 2.3). We show this below.

   Since the partition function is modular invariant,

$$\mathcal{Z}(\tau, \bar{\tau}) = \mathcal{Z}(-1/\tau, -1/\bar{\tau})\,. \tag{1}$$

   Using $\chi_j(-1/\tau) = \sum_{j'} S_{jj'}\chi_{j'}(\tau)$,

$$\bar{\chi}_i(-1/\bar{\tau}) M_{ij} \chi_j(-1/\tau) = \bar{\chi}_i(\bar{\tau}) S^{\dagger}_{ii'} M_{i'j'} S_{j'j} \chi_j(\tau)\,, \tag{2}$$

   and using (1),

$$\bar{\chi}_i(-1/\bar{\tau}) M_{ij} \chi_j(-1/\tau) = \bar{\chi}_i(\bar{\tau}) M_{ij} \chi_j(\tau)\,,$$

   we have,

$$S^{\dagger}_{ii'} M_{i'j'} S_{j'j} = M_{ij}\,. \tag{3}$$

   Only if $M_{ij} = \delta_{ij}$, the S-matrix is unitary. The matrix M denotes the number of modules contributing to a particular q-character which is a component of the vector-valued modular form χ. For example, $su(3)_1$ CFT has the modular invariant partition function,

$$\mathcal{Z}(\tau, \bar{\tau}) = |\chi_0|^2 + 2|\chi_3|^2\,. \tag{4}$$

   Generally, one can always 'unitarise' the S-matrix by writing the larger S-matrix [5].

   In the $su(3)_1$ example above, this would mean the 3×3 matrix S_u corresponding to the distinct characters in the partition function,

$$\mathcal{Z}(\tau, \bar{\tau}) = |\chi_0|^2 + |\chi_3|^2 + |\chi_{\bar{3}}|^2\,. \tag{5}$$

   The details of this example are given in example (iv) pg 493 of the journal of the paper [5].

8. *Immediately after, I didn't understand what was meant by "reduced S and T-modular matrices".*

   We borrowed the name reduced S-matrix from [6]. The reduced S and T-matrices act on the reduced or unflavoured characters. We will clarify this by rephrasing the paragraph below eqn (2.4),

   "We can only impose positivity of the Verlinde fusion coefficients as the additional requirement on the admissible characters if the S-matrix in eqn (2.4) is unitary [5, 7]. To put it a little differently, we generally do not expect the S and T-modular transformation matrices of eqn (2.4), which act on the unflavoured characters (here onwards called the reduced S and T-modular transformation matrices as in [6]), to be compatible with the MTC structure, in particular, corresponding Verlinde fusion coefficients are not necessarily positive integers. However, a unitary S-matrix can be obtained from the reduced S-matrix following [5], which is by construction compatible with the MTC structure. In particular, the unitary S-matrix produces positive integer Verlinde fusion coefficients,"

9. *I also didn't understand "a unitary S-matrix implies a 1-1 correspondence of the unflavoured characters with the modules". What could unitarity possibly have to do with the linear independence of q-characters?*

   If more than one module contributes the same unflavoured character, then the S-matrix is not unitary (3). However, if each module contributes a unique q-character then (3) implies that S is unitary since $M_{ij} = \delta_{ij}$.

10. *After (2.7), it is claimed that the spectrum is invariant under $j \to -1-j$. But, this is false as (2.9) notes.*

    We will correct the statement to,

    "The energy (L_0) spectrum remains invariant under the transformation, $\sigma_0 : j \mapsto -1-j$."

11. *The last sentence in Sec. 2.1 is baffling, especially as it references [17]. Shouldn't a q-expansion (about $q = 0$) always give the degeneracies (multiplicities) of weights in a module? The subtle discussion in [17] concerns expansions in ω (or z) which is much trickier.*

    We understand that the last line in section 2.1 needs to be clarified. However, we plan to remove this sentence since it doesn't contribute anything new.

12. *Is there a typo in (2.20)?*

    Yes. There is a typo in eqns (2.19) and (2.20). (2.19) should read,

    $$\chi^{\pm}_{j(n,k)}(\tau, z) = \chi_{j(n,k)}(\tau, z) \mp \chi_{j(\bar{n},\bar{k})}(\tau, z)\,, \tag{6}$$

    and (2.20) should read,

    $$\chi^{+}_{j(\bar{n},\bar{k})}(\tau, z) = \chi_{j(\bar{n},\bar{k})}(\tau, z) - \chi_{j(n,k)}(\tau, z) = -\chi^{+}_{j(n,k)}(\tau, z)\,. \tag{7}$$

13. *I guess (2.23) only holds for $k \neq 0$?*

    True, it only holds for $k \neq 0$. We will add this.

14. *I also dislike the reference in the first paragraph of Sec. 2.3 to [17], seemingly in support of something "manifest" that I honestly find incomprehensible.*

    Following your comment and our earlier discussion, we plan to rephrase the paragraph.

    "In this subsection, we will analyse the q-series expansion of even characters χ^+ in the $\omega \to 1$ limit. Since we have eliminated the pole at $\omega = q^0 = 1$, the even characters have a well-defined q-expansion in the region $|q| < 1$ in the limit $\omega \to 1$, with finite integer coefficients which are also positive for specific cases. Let us now focus on the unflavoured limit of the even characters."

15. *In Sec. 3.1, it may be worth noting that the term "threshold level" also has the name "boundary level" in the literature. I believe it was introduced by Kac-Roan-Wakimoto in their 2003 article, but it could have been earlier.*

    The comment would be included in section 3.1.

    "Among the special levels mentioned at the beginning of this section, the set with $p = 2$ saturates the admissibility bound. For this reason, the $\widehat{su}(2)$ levels are called threshold levels. They are also called boundary levels for the same reason. The term was probably first used in [8]."

16. *I'll add that it is somewhat dangerous to identify a unitary CFT from its modular data. There are many known examples where different RCFTs give the same representation of the modular group. I would not be surprised if there were also examples of inequivalent RCFTs whose q-characters matched!*

    We admit that the modular data alone is insufficient to fully characterise the CFT. We will add a statement in the conclusions at the end of the first paragraph to notify the reader of our shortcomings.

    "However, note that the modular data alone is not enough to classify the tentative RCFT completely. In other words, although we have introduced discrete flavour fugacity for the $\widehat{su}(2)$ sectors, we have ignored the flavour fugacity while classifying the RCFT."

17. *In the conclusion, it is claimed that even characters are expanded in the region $|z| < 1$ and $|q| < 1$ and that the radius of convergence is the same as that for a RCFT. However, this can't be correct as there can be poles in $|z| < 1$ depending on τ, right?*

    We recognise the mistake in the radius of convergence of the even character. We will correct it to only $|q| < 1$. We would change the relevant part in the second paragraph.

Please let us know if there are further comments or suggestions regarding these points. We appreciate it. Thanks.

References

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