Response to referee #2

We would like to express our gratitude to the referee for their thorough review of the manuscript. We appreciate their valuable comments and suggestions. We largely agree with the referee's feedback, and as a result, we have made significant improvements to the paper, particularly in Section IV, to enhance its quantitative aspects. Below, we address each point raised by the referee and outline the corresponding changes made in the manuscript.

The manuscript presents a study of circuit (and geometric) complexity of various quantum algorithms and unitary evolutions where the initial and target states are potentially separated by a quantum phase transition. The purpose of the manuscript is to show that the crossing of the quantum critical point leads to complexity increasing with system size. While such divergence is studied systematically for the Fubini-Study complexity where the scaling (dominated but the neighborhood of the QCP) is essentially connected to that of the fidelity susceptibility, for the Nielsen complexity the computation is done numerically for adiabatic algorithms and for VQE. The qualitative statement obtained is that whenever the algorithm is "aware" of a QCP the associated complexity is seen to increase and diverge with system size.

I think the subject matter of this paper quite interesting though at present the study seems more a collection of semi-qualitative observations rather than a systematic study of the relation between complexity and QCP. The introduction, section 1.A, is very nice though it would help to expand the explanation a bit, in particular between Eq.(11) and (14) where it is explained a connection between the two concepts. In particular I think it would be better not to confuse the reader and maybe use more space but state clearly the problem at all stages: the Nielsen complexity is related to transformations between two fixed initial and target state, the FS is a distance between states belonging to a given parametric manifold, the relation is given by such and such.

We are glad to hear that the referee finds our work interesting. We also agree that it is important to discuss the differences between the two complexities. In the initial version, we dedicated a subsection exclusively to addressing the distinctions between the Nielsen and Fubini-Study approaches (Subsection 1.A.3) . In response to the referee's comment, we have further revised this section to provide clearer explanations. We hope that the updated discussion now better reflects the connection between the two approaches.

In the part concerning FS Complexity is not clear to me what new quantitative elements are brought in in this paper as compared to what was already know about fidelity susceptibilities though QCP (see e.g. the works of De Grandi, Gritsev and Polkovnikov a few years back). It

would be useful to state it clearly.

We acknowledge the referee's observation that our previous statement was somewhat misleading. We now clarify that once the relationship between the complexity derivative and the metric tensor is established, the scaling of the former can be directly derived from studies on the metric tensor, particularly those related to quantum phase transitions (QPTs). We have taken into consideration the referee's suggestion and searched for the referenced paper. We believe the referee is referring to the work titled "Quench dynamics near a quantum critical point: application to the sine-Gordon model." We have cited with other papers using the fidelity susceptibility.

In turn, the part concerning Nielsen complexity, which to my knowledge is novel, is extremely qualitative. Certainly it gives a hint towards interesting phenomenology but It would have been useful to have a study (even numerical) of scaling with systems size, how this is affected (or not) by the properties of the QCP, etc.

We appreciate the valuable feedback from the referee, and we agree that our initial version lacked a more quantitative discussion. Their criticism has helped us to significantly improve this section.

In response to the referee's comments, we have made major changes to Section IV and added Appendix C. In Section IV, we now discuss adiabatic preparation (IV. A) and the use of VQE (IV. B). Additionally, we have included both theoretical and numerical studies on complexity scaling with various parameters, such as the system size, the energy gap, and the model parameters. Appendix C contains a detailed numerical investigation of the dependence of complexity in adiabatic algorithms on the energy gap between the ground state and the first excited state, validating the theoretical relationships discussed in Section IV A1.

Apart from rewriting Section IV, we have added a comprehensive summary of our quantitative study and its interpretation in the new version of the subsection "Main Results and Manuscript Organization" Section IB. This summary is provided in this response for the convenience of the referee:

" ... After this general discussion, we focus on calculating the complexity when preparing a fundamental state in a quantum computer. Here, obviously, we compute $\mathcal{C}_{\mathrm{N}}$ in its discrete version. We explore two algorithms in detail. First, we discuss the circuit complexity in adiabatic algorithms with and without shortcuts to adiabaticity. We focused our study on one-dimensional spin lattices of different sizes. In this investigation, we found that using shortcuts does not significantly alter the complexity $\mathcal{C}_{\mathrm{N}}$. However, we demonstrated that $\mathcal{C}_{\mathrm{N}} \sim \sqrt{L} \times T$, where L represents the system size, and T is the total time required to achieve a fixed fidelity, $\mathcal{F}$, with the exact ground state (in our case, $\mathcal{F} = 0.9$ was chosen). Thus, the complexity inherits the behavior of T close to a Quantum Phase Transition (QPT). Specifically, T is bounded by Δ^{-2}, where Δ represents the minimum gap between the ground state and the first excited state in the adiabatic algorithm.

Then, we discuss the circuit complexity using VQEs. These algorithms are variational and do not need to cross the critical point even if the reference and target are in different phases. In such a case, $\mathcal{C}_{\mathrm{N}}$ is not necessarily aware of the QPT. On the other hand, if the target state is close enough to a phase transition, also in VQEs, the complexity grows. Importantly, we provide an explicit formula for $\mathcal{C}_{\mathrm{N}}$, and by combining it with the correlation length generated using local Variational Quantum Eigensolver (VQE) ansatzs, we can show that $\mathcal{C}_{\mathrm{N}} \gtrsim L^{3/2}$. Therefore, this scaling poses challenges for our numerical capabilities, explaining the difficulties in finding reliable solutions around Quantum Phase Transitions (QPTs) when simulating the action of a VQE."

We hope that the referee may find the added discussions and results relevant and enriching to the previous version.

Overall I think that the paper with significant improvements in the presentation, in discussing the relation with previous literature and more quantitative statements in the final part could be published in SciPost Physics.

With all this, we thank the referee again for his helpful suggestions and comments and hope that the new version of our work will be to the referee's liking.