
Response to Anonymous Report 1

This work presents a detailed study of the vortex phase diagram of a conventional type-II superconductor. Due to the sizes considered, the self-consistent nature of the approach and physical quantities studied, it provides a significant improvement on previous results and identifies and characterizes different phases. Particular attention is paid on the effects of magnetic flux and disorder on the changes in the vortex lattice structure and the distortions on the vortices. The work is an interesting addition to the understanding of a disordered superconductor in a magnetic field and should be published.

Our response:
We thank the referee for his/her time and effort in reviewing our manuscript and for his/her positive and constructive report.

Questions/remarks:

1- Can the authors estimate for which size do the results improve and differ significantly from previous results in the literature?

Our response:
This estimation depends on what observable is computed and also on the magnitude of the electron-phonon interaction, the disorder strength and the magnetic field. For instance, if the goal is to determine the precise structure of the vortex lattice, it is necessary to be able to produce a sufficiently large number of vortices still within the superconducting region. The weak coupling region |U|< 1.25 requires at least a system size $N = 50 \times 50$, because vortices are larger, which was beyond the state of the art before our paper. This is in the clean or almost clean case. If disorder becomes stronger, the lattice will be deformed which would require an even larger number of vortices, and therefore larger sizes $N = 100 \times 100$ to characterize it. In general, the previous results in the literature only allow to study the strong coupling limit U>> 1 for which disorder effects are largely small at least in range of disorder strength that we are interested in. However, such strong coupling limit U>>1 is not really realistic as in most conventional superconductors the coupling strength, in dimensionless unit, is at most one. Therefore, our results reach a region of parameters close to the one that can be tested experimentally.

What is more relevant to the improvement of this study: larger systems or the self-consistency+ study of lattice deformation, etc?

Our response:
As we mention earlier, larger system sizes (50×50 or larger) are a necessary condition to study the vortex lattice structure in the relatively weak-coupling limit

typical of metallic superconductors such as Nb, Sn or Al. At the same time, a quantitative description of the effect of disorder, especially beyond the weak disorder limit, does require the consideration of the self-consistence condition. For instance, the spatial distribution of the order becomes broad even well in the metallic side of the transition. The selfconsistent condition is fundamental to reproduce theoretically this experimental result. Therefore, both the description of the deformation of the vortex core for sufficiently large disorder and the vortex lattice distribution for weaker disorder necessitates the use of the self-consistent condition as well.
In summary, it is safe to say that larger lattice sizes are a necessary condition for the determination of the vortex lattice structure in the weak coupling limit while the self-consistent condition, together with large sizes, are a requirement for the study of both the vortex lattice structure and the reported inhomogeneity of the vortex core if disorder is sufficiently strong.

2- The authors only consider the effect of the vector potential and neglect (as is standard practice) the effect of Zeeman term (coupling of the spins to the external magnetic field). If the magnetic fields are large, the spin coupling may have a significant effect, unless, for instance, the g-factor is small. It would be worthwhile to comment on the approximation of ignoring the Zeeman coupling.

Our response:
Yes, we agree with the referee that the Zeeman term is important if the magnetic fields are large. However, the large magnetic fields limit required for the Zeeman effect to be relevant is beyond the scope of the paper, since we are interested in the study of vortices in the still superconducting phase for which the magnetic field cannot be very strong because otherwise superconductivity will break down. As mentioned in the manuscript, it is increasingly difficult to identify the position and shape of vortices in the superconducting region when the magnetic field increases. Therefore, we expect that the consideration of the Zeeman term would not change qualitatively the main results of the paper.
Following the referee suggestion, in the updated version of the manuscript, we have added a few lines in the first paragraph on page 8 explaining in more detail why the Zeeman term was neglected in our analysis.

3- How is the stiffness calculated explicitly in magnetic field? It would be useful if some detail of the calculation is presented.

Our response:
We thank the referee for this suggestion. The detailed calculation of the superfluid stiffness was already presented in our previous papers [Physical Review B 105.9 (2022): 094515. Physical Review Letters 130.4 (2023): 047001.]. Since we are to large extent following the same computation scheme for the superfluid stiffness in this paper, we do not think a full repetition of the calculation is necessary. However, attending the referee request, we have added further details in the updated version

manuscript to facilitate the understanding of the stiffness calculation, see Appendix H. The second paragraph of page 26 is also modified accordingly.

4- Since the authors are able to work with larger systems, it would be interesting, in some future work, to consider gapless systems, such as d-wave superconductors.

Our response:
We thank the referee for this interesting suggestion. It is technically feasible to generalize our current code to study d-wave or p-wave superconductors, or even include the Zeeman term which makes the model more realistic. In fact, one of our current projects is the study of vortices in p-wave disordered superconductors.

Response to Anonymous Report 2

Strengths
1) Methodology: Authors use a microscopic model to investigate the interplay between disorder and vortex formation. For strong disorder this is superior to 'conventional' phenomenological approaches.
Considered system sizes are much larger than in previous investigations.

2) Careful and physically sound discussion on the various aspects of their findings.

3) Good introduction into the subject.

Weaknesses
1) At few places it is hard to correlate the discussion with the results shown in the figures.

2) Figure labeling

Report
In this paper authors investigate the interplay of disorder and vortex formation on the basis of an attractive Hubbard model with on-site disorder which is coupled to a magnetic field and solved within a Bogoliubov-de Gennes approach. Different regimes in the field-disorder phase space are identified. These comprise the conventional Abrikosov lattice in the small disorder regime, the transition toward a rectangular lattice at 'intermediate' fields, and the loss of translational invariance at

even higher fields. Also the superconducting properties as a function of the field are studied where it is found that up to intermediate disorder strengths the critical magnetic flux is enhanced. Moreover, for large magnetic fluxes disorder can even enhance the average superconducting order parameter.

This is an interesting paper which provides new insight into the actual and complex problem which makes a step forward to understand the influence of disorder on the vortex formation in superconductors. The paper is well written and meets the criteria for publication in SciPost.
I therefore recommend publication of the manuscript in SciPost after the points in "Requested changes" have been considered.

Our response:
We thank the referee for his/her time and effort in reviewing our manuscript and for his/her positive and constructive report. Some of the referee’s requested changes have certainly led to a better manuscript. Below is a detailed response to the referee’s comments and questions:

Requested changes
1.) According to Abrikosov theory the 'size' of the vortex core is determined by the coherence length. Despite that it is a central quantity in vortex physics the term 'coherence length' appears only once in the caption to Fig. 1. In my opinion it should be straightforward to evaluate the coherence length as a function of disorder (e.g. from the current-current correlations) and then compare with the vortex profile shown in Figs. 7-10.

Our response:
We agree with the referee that the coherence length is a central quantity in vortex physics, that it is straightforward to obtain it from the mentioned correlation function and that, in principle, a comparison with the vortex profile could be a natural check of our results. It is indeed a meaningful check but only for sufficiently weak disorder. Our results indicate that as disorder increases, the vortex profile is increasing determined by the details of the random potential while the superconducting coherence length is less sensitive to it. For instance, the vortex core becomes asymmetric and it is located in regions where disorder fluctuations heavily suppress the order parameter while the coherence length still reflects global properties of the superconductor. Precisely because in our case there is no in general a direct relation between the two observables we decided not to present explicit comparison.

However, after the referee’s comment, and given that many readers may not be familiars with all the details of the physics of disordered superconductors, we feel we should be more explicit about this point in the manuscript. For that purpose, in this update, we have included an explanation why the two quantities are in general different and therefore, the coherence length is not really suitable to characterize the vortex profile.

In order to support this statement, and for the sake of completeness, we present below an explicit comparison. We follow a previous study [PHYSICAL REVIEW B 92, 064512 (2015)] to extract the coherence length \xi_D from fitting the intrinsic superconducting response \Delta D_s(q_y) = D^{SC}_s(q_y) - D^{M}_s(q_y) = D_s [1-(\xi_D q_y)^2], where D^{SC}_s(q_y) is the superconducting component, and D^{M}_s(q_y) is the transverse current response of the normal state. \xi_D is the superconducting coherence length related to the current response. Here, we want first to note that in this calculation we didn't consider the vertex corrections, which can be important in the strong disorder limit around q_y \sim 0. We are not sure whether the concave shape in the strong disorder limit is due to absence of vertex corrections, which includes different fluctuation channels. Considering all vertex corrections in such large system is numerically demanding and not realistic with the computational resource we have. However, we expect that even including the vertex correction, the curvature does not change significantly.

The results are illustrated in Figure.R1. \xi_D decreases fast with disorder. In the clean limit, when V=0, \xi_D=14, which is similar to the vortex size r_0 = 12.9. In the weak disorder V=0.5, \xi_D=9.5, which is also consistent with the vortex size r_0 = 10.1. Note that r_0 is the radius of the vortices. However, when the disorder is stronger, **the SC coherence length \xi_D decreases significantly** (\xi_D = 4 when V=1), while **the vortex size decreases slightly**. Fig. 20 in the manuscript shows that r_0=9.35 when V=1.5. In this range of disorder, \xi_D <<r_0. When V>=1.5, the shape becomes concave around q_y \sim 0, which cannot be fitted with the parabolic prediction.

In Figure.R2, we have depicted D^{SC}_s(q_y) and D^{M}_s(q_y) under various disorder. Both show decreasing curvature with increasing disorder.

In Figure.R3, we tried to just fit D^{SC}_s(q_y) with formula D^{SC}_s(q_y) = D_s [1-(\xi_D q_y)^2] to obtain an approximate \xi_D in the strong disorder limit. The curvature is dominated by D_s \xi_D^2, which is smaller for strong disorder V=2.25.

We hope that those results provide evidence that the SC coherence length \xi_D <2 when V>=1.5, which is much smaller than the corresponding vortex size.

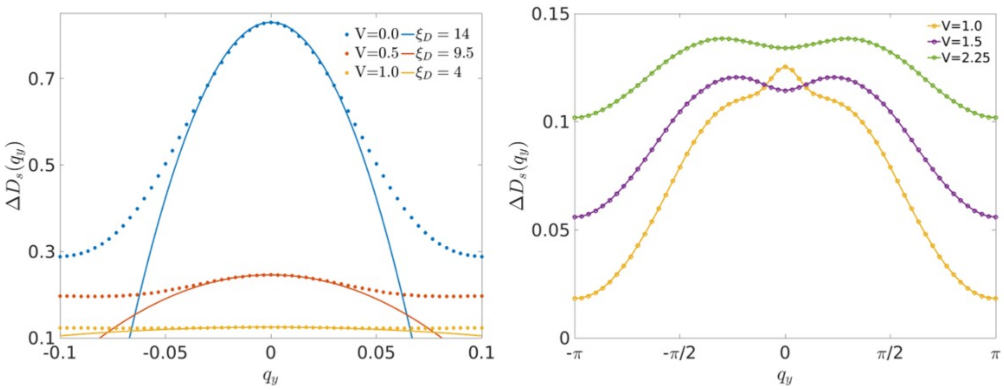

Figure.R1: The intrinsic superconducting current response \Delta D_s(q_y) as a function of disorder. Left panel: Only shows the results for small q_y, which can be fit to obtain the SC coherence length \xi_D. Right panel: \Delta D_s(q_y) with respect to the stronger disorder. The system size is N=60\times 60. The results are averaged over five samples when V>=1, but only one sample when V<=0.5. The other parameters are |U|= 1, <n> = 0.875, the magnetic flux \phi/\phi_0 = 0. The results in Figure.R2 and Figure.R3 are the same configurations.

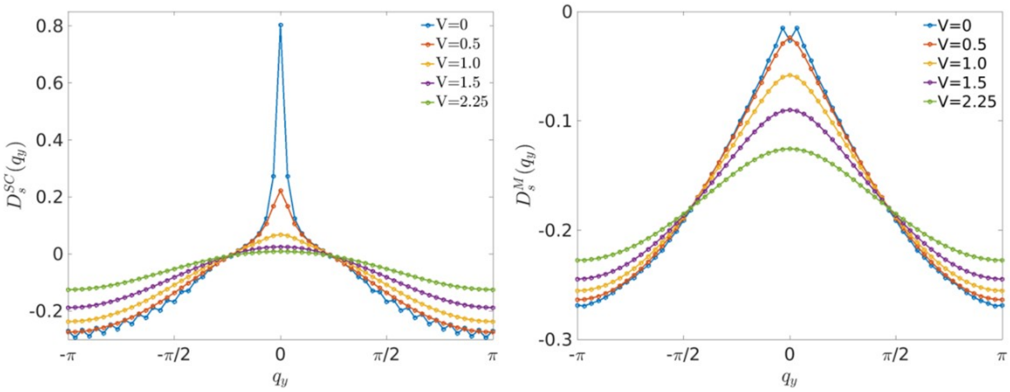

Figure.R2: The transverse current response for the superconducting system D^{SC}_s(q_y) (Left panel) and the normal metal system D^{M}_s(q_y) (Right panel)

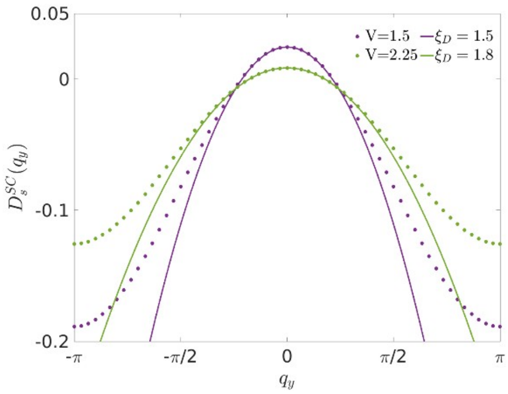

Figure.R3: The transverse current response for the superconducting system D^{SC}_s(q_y) and the corresponding fitted \xi_D.

2.) For the clean system the vortex lattice is only shown for values of the flux up to \phi/\phi_0=18. It would strengthen the discussion when authors would add to Fig. 3 a row with V=0. In fact, Fig. 1 seems to indicate that there is also a transition to a rectangular structure for V=0 whereas on page 14 (2nd row) it is claimed that this structure results from a compromise between disorder and magnetic flux. The question is therefore, whether for the clean case the lattice stays triangular up to high fields.

Our response:
Yes, we agree with the referee. However, in the clean limit, the vortex lattice breaks down quickly. We can see from Fig. 19 in Appendix C in the updated manuscript that superconductivity breaks down when the flux is 14. So, it is not possible to increase

the flux much more than we did in Fig.3 in the manuscript. However, following the referee comment, we feel we didn’t explain sufficiently well in Fig. 1 that for V=0, there is no such transition from triangular to rectangular. This has been corrected in the updated manuscript.

We cannot rule out that tuning the shape of the sample and the number and size of the vortices that a rectangular lattice is observed. However, it may require some fine tuning so it would not be a generic transition.

By contrast, in the weak disorder limit, even for different system sizes and a wide range of flux number, the transition is always observed, see Fig. 24 and Fig. 25. We think this transition is robust in this case because it results from the combined effect of disorder and the long-range magnetic interactions between vortices.

In the clean limit, since the size is finite and the system is symmetric, only configurations with a certain number of vortices respect the symmetry. Therefore, when there are 8 or 10 vortices in a square lattice, it would be of course impossible to form a perfect Abrikosov vortex lattice. When there are 12 vortices, a compressed Abrikosov lattice is reproduced, see Appendix C in the updated manuscript. That’s why we only consider the size N=100x100 in the main text in order to reproduce the Abrikosov lattice. We also add some results with system size N=60xW, where W varies. For our choice of parameters, the Abrikosov lattice is also well reproduced, although in some cases, depending on the value of W, the Abrikosov lattice is compressed or stretched.

3.) Page 12, last paragraph: The quantity \xi_0 is introduced as the vortex separation in the clean limit. I don't understand this definition because the vortex separation should depend on the flux. Does \xi_0=12 refer to the same flux where the rectangular lattice is observed? Please clarify!

Our response:
We agree with the referee that the discussion about \xi_0 was confusing. \xi_0 here means the vortex size, which is also close to the coherence length in the clean or weak disorder limit. The vortex separation is represented by L_v. What we meant is that when the vortex separation L_v is close to \xi_0, which means that vortices start overlap each other, the triangular-rectangular transition happens. We have rewritten the corresponding paragraph and the caption of Fig.5(b) in order to convey this idea more clearly.

4.) page 17, 2nd paragraph: "It is expected that the profile of the order parameter should match with the magnetic field inside the vortex.....". This statement and the following is misleading. The profile of the order parameter is determined by the coherence length whereas the decay of the magnetic field is ruled by the penetration depth and the functional forms of both quantities do not necessarily coincide. Eq. 4 is rather an Ansatz which allows to fit the order parameter profile but I would not relate this to a functional form for the magnetic field.

Our response:
We agree completely with the referee that the profile of the order parameter and the decay of the magnetic field are two completely different things. The Ginzburg-Landau theory predicts that the vortex profile is $\Delta(r) \sim \tanh(r)$ in the absence of disorder. In the updated manuscript, we have changed the theoretical model accordingly and updated the comparison with the numerical results in the updated manuscript. Although the functional form is different, the agreement between the numerical and the theoretical prediction is very good and similar to the one achieved with the previous expression.

5.) page 16: The correct limits for the definition of the superfluid stiffness are \omega=0 and the transverse momentum q->0. At the end of the same paragraph it is stated that (for V=2.25) "the superfluid stiffness becomes zero for a much smaller field strength \phi/\phi_0=16." However, Fig. 13a still reveals a finite stiffness of D_s=10^-3 - 10^-2 for this value of the flux.

Our response:
Yes, we agree with the referee that for V=2.25 there is still a finite stiffness D_s \sim 10^-3 when \phi/\phi_0=16 in Fig.13. However, the method of computing D_s in our case is not very accurate to predict the exact location of the phase transition though a sharp drop by a small change of disorder or flux is a rather precise indication of the critical region close to the transition. Moreover, we didn't consider the quantum phase fluctuations, which further suppresses the stiffness. As a consequence, the very small value of D_s for \phi/\phi_0=16 and V=2.25 likely signals that the superfluid stiffness is already zero.
Another source of uncertainty is that our mean field results are less reliable in this region of strong disorder and relatively large magnetic field. In the updated manuscript, we add some details about the range of applicability of our method of computing D_s in order to illustrate the limitations of the strong disorder (V = 2.25) results which are likely at the transition or already in the insulating region.

6.) Fig. 12: Why the correlation function is not periodic? Does the plotted r-range correspond to half of the lattice size?

Our response:
We thank the referee for raising this issue. Indeed, the correlation function was not properly defined in the manuscript. Since the disorder is not periodic, we didn't consider the periodicity when we calculate the correlation function. When we calculate the correlation function, only the sites with a specific distance r from the

chosen site are considered. We then do average over all sites to get the final correlation function. We have provided a precise definition of the correlation function in the updated manuscript so that it is clear now why the correlation function is not periodic.

Minor issues:

a) Eq. 1: Either the hamiltonian is defined for arbitrary hopping parameters, then one should replace -t-> t_{ij}. Or one introduces nearest-neighbor hopping from the beginning. Then this should be indicated in the sum over "i" and "j".

Our response:
We thank the referee for this suggestion. We only consider the nearest-neighbor hopping, so we replaced t_{ij} with t. In the updated manuscript, we clarified this issue in the very definition Hamiltonian and in the text around it as well.

b) Eq. 3: Replace t_{ij} -> t_{i,i+\delta} and put it under the sum.

Our response:
We thank the referee for spotting this typo. It has been corrected in the updated manuscript.

c) The results in Sec. IV are for 60x60 lattices. This is only specified at the end of Sec. IV but should be already defined at the beginning.

Our response:
We thank the referee for this suggestion. In the updated manuscript we state at the beginning of the Sec. IV that all results are for 60\times60 lattices. Moreover, we explain explicitly the different parameters that we use in the paper and justify the choice of parameters.

d) In all figures which report the Fourier transform the range of momenta should be indicated.

Our response:
We agree with the referee. We have improved the figures of the Fourier transform.