Authors' response to Referee #2

Strength:

1. *The technical computations are clearly exposed and easy to follow.*
2. *The studies of degeneracies in non-Hermitian systems has clearly become very relevant for experiments. This paper brings a clearer mathematical understanding to the nuances of their properties.*
3. *A formal characterization of the NDEP would be appreciated.*

Authors' response: We thank the Referee for highlighting the strength of our work and for finding its technical parts clearly exposed and easy to follow.

Weakness:

1. *In my opinion, the paper fails to properly introduce the different types of degeneracies in non-Hermitian systems. A short,mathematical definition (with an example for each) in Introduction or Appendix would greatly simplify the reading of the paper. I , in particular, would stress the importance of a more explicit definition of defective vs non-defective exceptional points early in the paper (it is barely half a sentence currently).*
2. *Similarly, the authors fail to emphasize the relevance of the distinction between defective and non-defective EP, in terms of physics.*

Authors' response:

We thank the Referee for bringing these weaknesses to our attention, which we tried to rectify in the revised version of our manuscript.

1) To address this point, we have added a schematic figure to our manuscript. In this figure for each type of NH degeneracies, we have summarized the associated constraints for their emergence, an intuitive way of identifying these degeneracies, and an example of such band touching points in two-band systems. We have also added a list in Section 2, where the various NH degeneracies are formally introduced and explained. We hope that the Referee finds this clarifying.

2) We thank the Referee for this important comment. We agree with the Referee that while mentioning the physical significance of our results is important, we would like to emphasize that with this work, we aim to demonstrate the mathematical distinction between different degeneracies in NH models. With respect to these points, aside from our formulations, we mention platforms where our findings can be witnessed. In the current manuscript, we also added,

> "The microwave experiments with a metallic mesh 3D photonic crystals have also realized $\mathcal{PT}$-symmetric models [67]. Here, defective EPs form chains, and the intersections of these EP lines represent non-defective EPs. These non-defective degeneracies in these particular experiments are sometimes protected by additional mirror symmetries; see Refs. [68,69] for details."

to provide another platform to realize part of our results. In addition, to address the suggestion of the Referee, we now point out the importance of non-Hermitian degeneracies in anomalous transport responses. We have added, "Aside from realizing NH degeneracies, the occurrence of these degeneracies in the spectrum may give rise to exotic responses. The NH anomalous currents observed in odd spatial dimensions exemplify these interesting responses. It has been shown that NH (non)interacting systems with ONPs, when coupled to gauge fields, e.g., electromagnetic fields, exhibit anomalous currents different from their Hermitian analog [5,70,71]. For instance, the NH chiral magnetic effect, in contrast to its Hermitian counterpart, may find room to emerge in equilibrium in $\mathcal{PT}$-symmetric systems [70]."

Authors' response to Referee $\#2$

The paper "Symmetry-protected exceptional and nodal points in non-Hermitian systems" gives generic criterion of existence for defective and non-defective high-dimensional exceptional points in general n bands models. Given the relevance of the non-Hermitian descriptions of experiments and the possibilities offered by exceptional points. the manuscript appears to be relevant to a general public.
While the paper is globally well presented, it is hard to follow, especially for non specialists. I also think there are several important typos. Consequently, I would only recommend the publication in SciPost after modifications.
Given the mistakes I found, I also strongly recommend a careful rereading of the manuscript as it is not impossible I missed others.

Authors' response:

We again thank the Referee for finding our work to be relevant to a general public and globally well presented. We also thank the Referee for spotting typos, which we have eliminated in the revised version of our manuscript.

I had a few questions, in addition to the changes I would like to see listed below.
1- Non-defective exceptional point are characterized as diagonalizable degenerate points in the neighborhood of which defective EP exists. Is there a form of (topological) invariant/signature one can derive to characterize them instead?
2- These NDEP split two manifolds/lines of DEP. Given this structure, a property should abruptly change when following these lines of DEP through the NDEP. At first sight, I expect that some properties of the defective eigenstates dramatically change (typically, handedness in the example you have) . Is this intuition correct and general? Can you take advantage of that to explain the resilience of the NDEP in high dimensions?

Authors' response:

1) We thank the Referee for raising this interesting question. The NDEPs are, as we mentioned in several places, stable in the sense that they are protected by the present symmetry. Therefore, it should indeed be possible to strengthen this claim (which is supported by counting degrees of freedom when solving for the trivial solution of the eigenvalue degeneracies) with the calculation of some kind of invariant. It is indeed possible to encircle/wind non-trivially around the NDEPs, making their classification in $\mathcal{PT}$-symmetric systems similar to that of Weyl points in Hermitian band structures. Importantly, we also notice that NDEPs in $\mathcal{PT}$-symmetric systems differ from those appearing in TRS†-invariant systems. NDEPs in $\mathcal{PT}$-symmetric systems can, just as Weyl points, be annihilated by merging two NDEPs of different "charge", while NDEPs in TRS†-invariant systems are stationary in momentum space and can therefore not be merged. Nevertheless, both kinds of NDEPs can be classified by calculating the winding/Chern number around the singularity, as they can be encircled by non-trivial loops. We have added

sentences explaining this in Sec. 2.

2) We thank the Referee for sharing this nice intuition. NDEPs reside along the manifolds/lines of the DEP at which the corresponding (traceless part of the) Hamiltonian becomes null. As the Referee correctly identified, in our examples, the NDEPs are located in the middle of two DEPs at $\tilde{\mathbf{k}} = 0$, and hence, the change in the handedness can be a practical measure to identify them. However, we cannot characterize specific mathematical constraints that rule out the possibility of finding not-equally distanced NDEPs from their surrounding DEPs.

Regarding the stabilities of NDEPs in general, the resilience of symmetry-protected NDEPs in any dimension depends on the stability of their underlying symmetry. As long as these symmetries hold, NDEP is robust against perturbations.

> *1- Give concrete and explicit mathematical definitions of the different type of degeneracies discussed in the manuscript, preferably in Section II (or in an Appendix for examples) and stressing the differences.*
> *2-End of page 4, you discuss TRS†. You claim that "it enforces all symmetric parts of d to be 0". It does not seem to be the case for d_{xI}, d_{yR} and d_{zI} which should be symmetric. Following that it seems that the corresponding result in the table is incorrect.*
> *3- The limits in Eq. 15 appear ill defined: the limit on k_y is only well defined if $k_z^2 > k_x^2$ and then k_z is sent to 0 while k_x is kept finite. The limit on k_z does not appear to be necessary to show the desired property: fixing $k_y^2 = k_z^2 - k_x^2$ is enough.*
> *4- Fig. 2a: the spectrum is not symmetric under $k_x \to -k_x$. Given the form and the parameter, there seems to be a mistake.*
> *5- Could you clarify the reason of the double degeneracy in your 4 band model.*
> *6- The spectrum in Fig. 4. Could you develop/clarify and illustrate in Section 4 the experimental signature of the presence of non-defective EP*

Authors' response:

1) We thank the Referee for pointing out this lack of clarity, as flawlessly presenting our newly introduced concepts is of high importance. Following the Referee's suggestion, we have now included a schematic figure to briefly summarize the findings of our work on the differences between various NH degeneracies and how to detect them. In connection with the text following Eq. (11) in the updated manuscript, we have also added a list where more formal definitions and requirements discriminating the different degeneracies are summarized. We hope these modifications clarify the differences between the concepts in a manner that is satisfactory to the Referee.

2) We thank the Referee for pointing this out, as the confusion was sourced by a typo in the previous version of the manuscript. There, we wrote that Hamiltonians satisfying TRS† are of the form $\mathcal{H}(\mathbf{k}) = C_+\mathcal{H}(-\mathbf{k})^\dagger C_+^\dagger$, which is not correct. Instead of transforming with its Hermitian conjugate, the Hamiltonian should transform with its transpose, and hence Hamiltonians satisfying $\mathcal{H}(\mathbf{k}) = C_+\mathcal{H}(-\mathbf{k})^\intercal C_+^\dagger$ are the TRS†-symmetric ones. We have corrected this in the updated version of the manuscript.

3) We thank the Referee for this observation. In the updated manuscript, we instead consider a different limit where the previous inconsistency is not present. Indeed, the eigenvectors coalesce already in the first limit. The important thing to note with this reasoning is that they coalesce at the non-defective EP when approached from the directions of the defective EPs, while they do not coalesce if one approaches the non-defective EPs in a direction other than along the defective EPs. This means that the non-defective EP cannot be thought of as a conventional Hermitian ONP nor as a conventional non-Hermitian defective EP. We hope that

the updated version of the manuscript clarifies this.

4) We thank the Referee for this crucial observation, which was indeed due to a mistake. We have updated the figure with the correct spectrum.

5) We thank the Referee for this comment, as the interpretation that our model Eq. (19) is fully doubly degenerate is due to a lack of clarity in the description of the model. In fact, the period model is not fully doubly degenerate. Instead, it displays a double degeneracy along certain cuts in momentum space. The cut $k_x = k_y$ is one such and is what is displayed in Fig. (4). As this particular cut simplifies the illustration of the appearance of the non-defective EPs, we have chosen to display the band structure along this cut. We have clarified this in the caption of Fig. 4. It should also be stressed that the bands are doubly degenerate when linearized around $\mathbf{k} = (0, 0, k_0)$, as is pointed out right after Eq. (22). We emphasize that this is nothing that affects our general findings but rather helps us illustrate them in a simplified fashion. We believe this should now be clear in the updated version of the manuscript.

6) Fig. 4 in the previous version of the manuscript, which in the updated manuscript corresponds to Fig. 5, shows the spectrum of an NH system hosting defective EPs and ONPs. There are no non-defective EPs present in this spectrum. It is important to note that the ONPs present in these systems are not necessarily stable—they occur as a direct consequence of fine-tuning. Infinitesimal, yet finite, perturbations can make them disappear. Due to the highly fine-tuned nature of this spectrum, we do not expect any novel experimental signatures arising from these kinds of systems. This model is included simply to show that there exist non-defective degeneracies that differ from non-defective EPs and, thus, to highlight the different theoretical natures of the non-defective degeneracies. We have clarified this in the updated manuscript by adding "... and therefore these systems may not exhibit experimental signature different from those in generic NH systems" in Section 3, right before introducing the example model. We thank the Referee for raising this point and hope the clarification is satisfactory.

Minor points
7- Fig. 2 and 3 should be made larger. In both cases, c) and d) are barely understandable. Given that both Figs describe traceless models with only 2 effective bands (with the double degeneracy for Fig. 4), I would recommend plotting a pcolor map (or something similar) of one of the bands only for all of these graphs. If the authors want to stress square root profiles (or other), then showing a cut would probably be enough.
8- Eq. 17 and 18 are valid for $\tilde{k}_y = \sqrt{k_z^2 - k_x^2}$. It is not very clearly specified. Also, I seem to find the denominator to be $\tilde{k}_x + \tilde{k}_z$, though it might be a question of convention.
9- Eq. 21 and 22: I also have a minus sign in front of k_z.
10- Eq.23 and 24: The two limits are again not really necessary.

Authors' response:
7) We believe that the Referee is raising an important point here and have therefore updated Figs. 2 and 3 c) and d). We have swapped the previous surface plots for line plots along various cuts in momentum space, which is indeed sufficient to illustrate the desired phenomena and does so in a much clearer way. We hope that the Referee is satisfied with these changes.

8) We thank the Referee for pointing out this important detail. In the previous manuscript we indeed mention that the limit is valid when approaching the non-defective EP along the EPs. It was, however, not specified along which part of the disconnected sheets of defective EPs the non-defective EP was approached. This has now been specified in the updated version of the manuscript. Additionally, the Referee correctly spotted a sign error in the Jordan decomposi-

tion. The expression we used was indeed valid for the point $k_z = -\pi/2$, and not $k_z = \pi/2$. We have now corrected this mistake and thus get the same sign convention as the Referee. It should, however, be noted that our conclusions remain the same. We thank the Referee for this detailed observation and for spotting this mistake.

9) This sign error is also corrected, in line with the answer above.

10) As we want to show the behavior around the non-defective EP, both limits are necessary. In fact, one should eventually take all momentum components to 0, and it is exactly this limit that is of interest. We have clarified this in the updated version of the manuscript and thank the Referee for bringing this concern up.

Summary of changes:

- We have added

  > “The microwave experiments with a metallic mesh 3D photonic crystals have also realized $\mathcal{PT}$-symmetric models [67]. Here, defective EPs form chains, and their intersection of these EP lines represent non-defective EPs. These non-defective degeneracies in these particular experiments are sometimes protected by additional mirror symmetries; see Refs. [68,69] for details.”,

  > ”Aside from realizing NH degeneracies, the occurrence of these degeneracies in the spectrum may give rise to exotic responses. The NH anomalous currents observed in odd spatial dimensions exemplify these interesting responses. It has been shown that NH (non)interacting systems with ONPs, when coupled to gauge fields, e.g., electromagnetic fields, exhibit anomalous currents different from their Hermitian analog [5,70,71]. For instance, the NH chiral magnetic effect, in contrast to its Hermitian counterpart, may find room to emerge in equilibrium in $\mathcal{PT}$-symmetric systems [70].”

  and

  > “In addition to the setup mentioned above, numerous studies on heterostructures report the occurrence of exceptional points in these systems. However, to the best of our knowledge, no record of non-defective degeneracies is reported in these systems [6, 67-69].

  in section 4.

- Figures 2 and 3 in the previous version of the manuscript (Figures 3 and 4 in the updated manuscript) have been updated.
- The symmetry constraint defining TRS† has been corrected.
- A schematic figure explaining the differences between the various eigenvalue degeneracies has been added as a new Figure 1.
- A list including formal definitions of the various degeneracies has been added.
- Corrections of minor typos throughout the manuscript.
- Corrections of sign mistakes in Eqs. (17), (18), (21) and (22).
- Change of limits in Eq. (15).
- Addition of discussion on topological invariants for non-defective EPs.
- Expansion of the reference list as mentioned in the above answer.