Response to Referee Reports for the Manuscript: "Traveling/non-traveling phase transition and non-ergodic properties in the random transverse-field Ising model on the Cayley tree"

(Dated: 2023-09-21)

Dear Editor,

We are resubmitting our revised manuscript, titled "Traveling/non-traveling phase transition and non-ergodic properties in the random transverse-field Ising model on the Cayley tree," for your consideration for publication in SciPost Physics. We appreciate the time taken by you and the reviewers in evaluating our work.

We are delighted that both reviewers found our manuscript interesting and valuable for the community. Referee 1's comments were particularly positive, strongly recommending its publication in SciPost Physics.

Referee 2 provided a very detailed report, raising important and interesting questions regarding the analogy between the random transverse field Ising model we consider and the Anderson transition, both on the Cayley tree. In response, we have provided detailed explanations and made revisions to the manuscript to clarify these points. We have also followed their advice to enhance the overall presentation of our manuscript.

Furthermore, we have meticulously reviewed the bibliography.

To facilitate your review process, we have highlighted all changes made to the main text in magenta within the manuscript.

We greatly appreciate your consideration of this resubmission and eagerly await your response.

Sincerely, the Authors.

I. RESPONSE TO REFEREE 2

We would like to express our gratitude to Referee 2 for providing a detailed report, posing interesting questions, and offering valuable suggestions. We have meticulously addressed all of them, and we believe that the manuscript has greatly benefited from the recommended changes. In the revised manuscript, we have included references recommended by the Referee. Our response refers to the reference numbers in the new manuscript version.

1. On of the main concerns is related to the fact that Jordan-Wigner transformation for the random transverse field Ising model in 1d is known to be mapped to the free-fermion system with some induced superconductivity (a quadratic Hamiltonian, conserving only particle-number parity). From the first glance, it seems that the same construction should be valid also for the Bethe lattice, as there are no loops on such graph and, thus, one can easily sort the sites in a certain 1d ordering. As soon as one can map the system to the non-interacting fermionic one, it seems natural to expect that the model will be in the same universality class as the Anderson model on the Bethe lattice. Please comment on whether it is the case and if so, what you predict about the critical exponents for the corresponding Anderson model on the Bethe lattice.

   Answer: On the Jordan-Wigner fermionization of the spin model on the Cayley tree: The no-loop requirement is not a sufficient condition for having a simple formulation of the original TFIM Hamiltonian in terms of free fermions. Indeed we will end up with a non-trivial and non-local phase term in both hopping and pairing terms (when the TFIM is FW fermionized). In the new reference [86] (theorie.physik.uni-muenchen.de/TMP/theses/riedthesis.pdf), it is shown that one cannot rewrite the TFIM as a non-interacting fermion Hamiltonian on the Cayley tree (see also Ref. [85]). However, the "interaction" generated by the non-local phase terms might be irrelevant such that one retrieves some common features between the Anderson problem and the TFIM. We explain the similarity of the behaviors observed in Anderson localization and the cavity mean field (CMF) approach to TFIM on the Cayley tree from a traveling-wave perspective in the next point.

   Changes made: We have included a paragraph in the introduction that discusses the Jordan-Wigner fermionization of the spin model on the Cayley tree in comparison to the 1D case.

2. If the authors still claim that their model is interacting and cannot be mapped to any non-interacting model, then there appears another question, related to the boundary conditions (bare regularizer B_0). A mentioned in [30] for the Rosenzweig-Porter model, by scaling $B_0 \sim N^{-\phi}$, with the number of sites on the graph, one can address not only Anderson localization transition, but also the ergodicity-breaking one. Why does the same scaling procedure work for the interacting model in focus?

   Answer: The relationship between the TFIM in the CMF approach and Anderson localization on the Cayley tree can be understood by drawing an analogy to the traveling wave phenomenon discussed in previous works (Refs. [23, 33, 70]) and our present paper. In Ref. [23], Monthus and Garel demonstrated that Anderson localization on the Cayley tree corresponds, in the localized phase, to a traveling wave problem similar to the one described in our paper for the disordered phase of the TFIM on the Cayley tree.

To elaborate further, the typical procedure for studying Anderson localization on the Cayley tree/Bethe lattice is as follows: (i) The exact recursion relation for the Cavity Green's function from Ref. [20] is derived. (ii) A self-consistent solution for the distribution of the cavity Green's function is sought by introducing a small imaginary part, η, to the energy, i.e., shifting $E \to E + i\eta$. This can be accomplished numerically using population dynamics or pool methods.

It should be noted that Refs. [23, 33] deviate from this standard approach. Instead, they only consider η at the boundary, specifically in the first pool, and analyze the response after a certain number of iterations. Under these conditions, a traveling wave regime emerges in the localized phase (instead of a self-consistent solution). Furthermore, our approach differs from Refs. [23, 33] in that we consider a finite Cayley tree without resorting to the pool method. This allows us to account for finite-size effects, both analytically and numerically, as discussed in detail in the manuscript below (see our reply to the point 5).

Nevertheless, the traveling wave regime of the Anderson localization problem, well described in the Appendix of Ref. [23], exhibits several distinctions from the TFIM traveling wave regime described in our paper. Firstly, the cavity Green's function of the Anderson model is a complex number, leading to two coupled recursions for its real and imaginary parts. In our case, the cavity mean field is real and governed by a single, simpler recursion. Secondly, the distribution of the imaginary part of the cavity Green's function displays a fat-tailed exponent, denoted as β in Ref. [23] (see Eq. (A10)), that varies with the disorder strength W in the Anderson problem, while the corresponding exponent γ in our paper (see our Eq. (11)) remains constant in the TFIM.

These two differences make our case significantly simpler to analyze analytically compared to the Anderson localization problem. In particular, the value of g_c and the tail exponent γ are analytically known in our case, providing us with numerous analytical predictions. As we explain, when $g > g_c$, two possibilities arise depending on the boundary condition. If B_0 is a constant, i.e., independent of the system volume N, a stationary regime for the distribution of B is reached when the tree's depth is large. This corresponds to the "ergodic" delocalized regime of the Anderson problem. Alternatively, if we consider a boundary condition $B_0 \sim N^{-\phi}$, a non-ergodic regime is found where $B_{\rm typ} \sim N^{D_1-1}$. This behavior can be understood simply through the traveling wave analogy presented in section 6 of our paper: the initial condition is such that the traveling wave never reaches the wall $B \approx g$, where the nonlinearity of the recursion generates a stationary ("ergodic") regime.

We anticipate a similar outcome in the Anderson problem. In fact, the inverse thermodynamic limit discussed by Kravtsov et al. in their reference [29] involves setting $\eta \sim N^{-\phi}$ at every site on the Bethe lattice. In contrast, our approach is comparable to applying $\eta \sim N^{-\phi}$ exclusively at the boundary (specifically, in the first pool).

To summarize, it is the analogy with the traveling wave problem which brings similar behaviors between the TFIM and the Anderson transition on the Cayley tree.

Changes made: We have included two new paragraphs in the introduction that discuss these analogies.

3. As a follow-up comment, I would like to emphasize that in [30] the scaling of the regularizer with N was important in order to find the scaling of the mini-band width (Thouless energy), which in the non-ergodic extended phase scaled down with a certain $\phi > 0$. At the same time it is known that in the Anderson model on Bethe lattice and/or on the random regular graph, the Thouless energy does not scale with N and stays finite (but small with respect to the bandwidth) up to the Anderson transition. Please comment on why do you need to take $B_0 \sim N^{-\phi}$? Do you expect to have the corresponding Thouless energy scaling down with N?

   Answer: The Referee is highlighting the distinction in nature between the delocalized phase of the Anderson transition on random regular graphs (RRG) and the infinite Bethe lattice/Cayley tree, which is ergodic, and a finite Cayley tree or Rosenzweig-Porter type random matrices where the delocalized phase is non-ergodic. In our current case, a similar differentiation could be expected on the finite/infinite Cayley tree. It essentially boils down to a matter of the order of limits. Due to the finite boundary fraction compared to the total volume on a finite Cayley tree, the order of limits $B_0 \to 0$ and $N \to \infty$ becomes crucial. In the usual order, where $N \to \infty$ first, followed by $B_0 \to 0$ (corresponding to the infinite Bethe lattice), the ordered phase exhibits ergodic behavior. Conversely, in the inverted order, where $B_0 \to 0$ precedes $N \to \infty$ (corresponding to $B_0 \sim N^{-\phi}$), the non-ergodic ordered phase emerges. It would be intriguing to explore the propagation of the cavity mean field equations of the TFIM on a random graph without a boundary but with loops, such as the RRG (refer to Ref. [10] for a similar approach concerning the Anderson problem), to verify whether the resulting ordered phase is ergodic.

Changes made: In the conclusion, we have added a paragraph discussing the differentiation between the Bethe lattice and the finite Cayley tree, as well as the intriguing perspective of Random Regular Graphs.

4. If the authors claim that the $B_0 \sim N^{\phi}$ in the non-ergodic extended phase is related to the N-scaling of the Thouless energy, they should consider either the overlap correlation function $K(\omega)$ or the local density of states, showing the corresponding miniband structure, like in the Rosenzweig-Porter.

   Answer: The CMF equations describe a non-trivial mean-field approximation of the zero temperature ground state of the TFIM. We are thus not dealing with localization properties of highly excited eigenstates. Hence, the notion of Thouless energy and overlap correlation function are not relevant, we believe, in our problem.

   Changes made: In the conclusion, we have included a paragraph to emphasize the limitations of the analogy between our cavity mean field approach and Anderson localization.

5. The random regular graph is known to have drastic finite-size effects, while in the current model (which seemed to be mapped to a very similar model) the authors seem to overcome this issue. Please comment on the finite-size effects in the model in focus and on the possible influence of them on the (numerical) results. Especially this question should be asked to the claimed presence of the non-ergodic extended phase. In order to clarify this, please show the drift of the extrapolated fractal dimension D_1 in Fig. 10, fitted from a sliding window over system sizes.

   Answer: This question holds particular significance in our study as one of our primary objectives is to address the impact of finite-size effects on this issue. To account for these effects, we deviate from the standard approach of using the pool method and instead consider a finite-size Cayley tree with a specific boundary condition. We then use a mapping to a traveling wave problem, enabling us to analytically deduce finite-size corrections. In the traveling wave regime, which corresponds to the disordered phase $g < g_c$ in the TFIM, we predict the existence of a universal logarithmic finite-size correction that solely depends on the tail exponent γ, as indicated in Eq. (9). By incorporating this logarithmic correction,

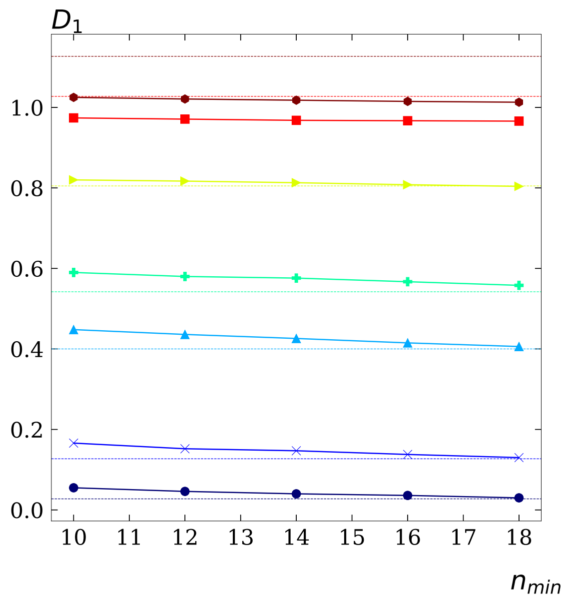

FIG. 1. The exponent D_1 obtained from a fit in a sliding window over system sizes $[n_{\rm min}, 20]$. It is shown as a function of the lower system size value of the window, $n_{\rm min}$, for each value of g considered in Fig. (10) (we use the same symbol as in the paper). The dotted lines correspond to the theoretical value obtained from Eq.(30).

we can accurately determine the velocity of the traveling wave, as depicted in Fig. 5(b). Importantly, the non-ergodic regime discussed in our paper corresponds to a traveling wave regime, whose velocity is directly associated with the exponent D_1, as outlined in Eq. (30) of section 6. We therefore use the universal logarithmic corrections to precisely evaluate D_1, as illustrated in Eq. (27). The numerical determination of D_1 exhibits excellent agreement with the analytical prediction, as shown in Fig. 10. In particular, incorporating the logarithmic correction leads to a fractal dimension D_1 which does not drift, as shown in Fig. 1 of this reply obtained by fits from a sliding window over system sizes.

Our study delves much deeper into the realm of finite-size corrections, extending beyond the traveling phase. Through the mapping to the traveling wave problem, we are able to make predictions regarding the intricate finite-size behavior at criticality, as evidenced in Eq. (23) and Figs. 6 and 7. It is worth noting that we have also verified the existence of a similar critical behavior for the Anderson transition on the Cayley tree, results which will be presented in a forthcoming paper. Additionally, the mapping technique grants us access to the complete single parameter scaling function that describes the critical properties and their finite-size scaling in proximity to the transition, including the values of the critical exponents. These predictions have been confirmed through meticulous finite-size scaling analysis, as depicted in Fig. 8.

Changes made: We have inserted a paragraph at the end of Section 5, 'Non-ergodic phase,' and in the conclusion, to address how our approach enables us to handle these intricate finite-size effects.

6. The Anderson model on the random regular graph is known to show the mobility edge behavior. Please comment on the presence of the mobility edge in the model in focus and, in the case of its presence, please clarify which averaging over the eigen energies has been taken to calculate fractal dimensions and other measures of the transitions. It might happen that the energy-resolved measures are needed for the case in focus. This is especially important, taking into account spatial inhomogeneity of the model, discussed in Sec. 7, as it can imply some spectral inhomogeneities as well.

   Answer: We consider in this paper the ground state properties of the TFIM in the CMF approximation. Therefore, we are not considering the localization properties of highly excited states. A mobility edge might be present in the excitation spectrum of the TFIM, as suggested in Ref. [70], section IV, but the analysis of this interesting property goes beyond the scope of our paper (the case corresponds to different recursion relations).

   Changes made: In the conclusion, we have included a paragraph to emphasize the limitations of the analogy between our cavity mean field approach and Anderson localization.

7. In Figure 3 deeply in the ordered (delocalized) phase, there is an apparent bimodal distribution P(lnB). Please comment on the origin of this bi-modality.

   Answer: The bimodal distribution has been discussed in Ref. [89], see section III (below equation (13)) and Appendix, below Eq. (A4).

   Changes made: We have included a comment in the current Figure 5 (formerly Figure 3).

8. In addition to the previous questions and comments, I would like to draw the authors' attention to different values of the branching number K, especially to the small-world networks, considered for the case of the Anderson model on the random regular graph by some of the authors of this manuscript. What do you expect to see for $1 < K < 2$ in the considered model? What are the peculiarities of this model?

   Answer: The tree graph corresponding to the smallworld model with $1 < K < 2$ has been described in Ref. [33], in the context of Anderson localization, see in particular the supplemental material, section "Recursion equations". More rencently, it was addressed in [99], see section 5.3 of that reference. Based on these two studies, we do not expect important qualitative changes with the case $K = 2$.

On the other hand, for graphs without boundaries and with loops such as random regular graphs and smallworld networks, we do expect significant changes. In particular, the non-ergodic ordered phase might not be present. This case could be addressed following the approach discussed in section IV of Ref. [32].

Changes made: In the conclusion, we have added a paragraph discussing the differentiation between the Bethe lattice and the finite Cayley tree, as well as the intriguing perspective of Random Regular Graphs and smallworld networks.

Minor changes:

(a) Please clarify in the abstract what "constant and algebraically vanishing boundary conditions" mean: it is unclear, while reading for the first time.

Changes made: We have changed the abstract clarifying what "constant and algebraically vanishing boundary conditions" mean.

(b) References [30] and [55] are mostly devoted to the Rosenzweig-Porter model, but not to the Bethe lattice or random regular graph: I am not sure that [30] is correctly cited in several places, as well as [55].

Changes made: We have removed the citations to Ref. [30] when exclusively discussing the Cayley tree case.

(c) The reference list on a non-ergodic delocalized phase are far from being complete: the works on Gaussian Rosenzweig-Porter model contain not only [30] and [55], but also - mathematical proof of it https://doi.org/10.1007/s11005-018-1131-7
- further investigations in statics
https://doi.org/10.1209/0295-5075/116/37002
https://doi.org/10.1103/PhysRevE.98.032139
and dynamics
https://doi.org/10.1088/1751-8121/aa77e1 - including subdiffusive behavior
https://doi.org/10.1209/0295-5075/117/30003

https://doi.org/10.21468/SciPostPhys.6.1.014

There are some (multifractal) generalizations of the Rosenzweig-Porter models with fat tailed distributions of off-diagonal elements:
starting from Levy-Rosenzweig-Porter:
https://doi.org/10.1088/1751-8121/aa77e1 - also mentioned above in the dynamics
https://doi.org/10.1103/PhysRevB.103.104205
-Log-normal Rosenzweig-Porter:
https://arxiv.org/abs/2002.02979
https://doi.org/10.1103/PhysRevResearch.2.043346 = [48]
https://doi.org/10.21468/SciPostPhys.11.2.045 - including the subdiffusive dynamics

Even in short-range Floquet-driven systems one can observe multifractal phases:
https://journals.aps.org/pre/abstract/10.1103/PhysRevE.81.066212
https://doi.org/10.1103/PhysRevE.97.010101
https://journals.aps.org/prb/abstract/10.1103/PhysRevB.103.184309
https://doi.org/10.21468/SciPostPhys.4.5.025
https://journals.aps.org/prb/abstract/10.1103/PhysRevB.93.104504
https://doi.org/10.1103/PhysRevB.106.L020201
https://scipost.org/SciPostPhys.12.3.082

In addition, in the correlated setting of the on-site disorder with short-range hopping, there is a whole bunch of works on Aubry-Andre model with p-wave superconducting pairing, showing a fractal phase. This wave has probably started with two works
http://dx.doi.org/10.1103/PhysRevLett.110.176403
http://dx.doi.org/10.1103/PhysRevLett.110.146404,
followed by the phase diagram calculation of the fractal phase in
https://journals.aps.org/prb/abstract/10.1103/PhysRevB.93.104504
and now has quite a number of publications (please see the works citing the latter one). Please consider to cite some of the representative papers in your work.

Answer: We thank the Referee for bringing these references to our attention. Following the Referee's advice, we have cited some of these representative papers.

Changes made: We have added a sentence to the second paragraph of the introduction, discussing the substantial body of work describing non-ergodic delocalized/multifractal phases.

(d) It is rather hard to go back and forth in reading the numerical part of the manuscript as it refers to the analytical part quite heavily. Please consider to re-arrange the manuscript in such a way to make it readable without massive back-and-forth scrolling.

Changes made: We have followed the Referee's advice and combined the previous sections 4 and 5 into a single section 4, where numerical results are discussed immediately after the corresponding theoretical predictions.

(e) The same is true about the location of the numerical figures: please place them in the corresponding places, where you discuss them, but not a couple of pages before.

Changes made: We have reorganized the placement of the figures, positioning them as closely as possible to the corresponding discussions in the text.

(f) The usage of the notion of the inverse participation ratio in (31) and Fig. 12 is very confusing as it is related usually to the fractal dimension D2. Please call I2 in (31) the second moment in order to avoid this confusion.

Changes made: We have replaced the term 'inverse participation ratio' with 'second moment' for I_2 in (31) and Fig. 12.