Response to Reviewer #1

The paper by G. Venditti et al addresses the important issue of inhomogeneous superconductivity realized in the low dimensional systems - specific examples of experimentally observed filamentary superconducting patterns are listed in the introductory section. Evidence for filamentary inhomogeneous nature can be manifested for instance by a broadening of the resistive transition or precursor gap observed above the critical temperature in tunneling spectra. The authors focus here on 2-dimensional case, analyzing the microwave transport properties within the 'random impedance network' (RIN) scenario. In particular, they distinguish characteristic features of the complex conductivity resulting from the filamentary structure of superconducting regions in the under- and over-doped limits, respectively. Ingredients of their theoretical approach are described in Section 3. In Sections 4 and 5 the authors analyze the complex microwave conductance over the temperature region, ranging from below to above Tc. They consider their calculations, confronting them with experimental data for $SrTiO_3$-based heterostructures. Global transition to the superconducting state is here driven by onset of the phase coherence, therefore the authors study in detail the superfluid stiffness that is encoded in the imaginary part of the complex conductance. Furthermore, they also point out some qualitative differences of the real part conductance near Tc which occur between under- and over-doped samples (as clearly displayed in Figure 1). The numerical results presented in Section 5 provide important information about the role played by geometry and disorder on the resistive and superfluid properties for the varying gate voltage. Such results illustrate the influence of filamentary superconducting condensate. The paper is clearly written and theoretical results are confronted with the experimental data. This approach might be useful to other highly inhomogeneous superconducting systems, therefore I would recommend the paper for publication.

We thank the referee for their very positive report and for recommending our paper for publication in SciPost.

Before any final decision, however, I kindly ask the authors for a few (rather technical) explanations/amendments.
(i) In Figure 2 the complex conductivity is presented for the microwave frequency 0.36 GHz. Is this particular choice of frequency representative for the considerations of resistive and superfluid properties?

We thank the referee for their comment which allowed us to clarify this point better.
The complex conductivity presented in Fig. 2 is experimentally measured at the circuit's resonant frequency, i.e. $\nu = \omega/2\pi = 0.36$ GHz. Indeed, when $\hbar\omega \ll \Delta$, the system behaves as a resonant circuit and one can access the superfluid stiffness J_s directly from its inductive response, being $J_s = \frac{\hbar^2}{4e^2 L_k}$, where L_k is the inductance of the circuit.

We included this clarification in the revised version of the manuscript.

(ii) Would it be feasible to obtain the frequency dependent σ_1 and σ_2 using RIN technique? If so, perhaps the authors could present some typical plot, showing the complex conductance within the frequency region from $\omega = 0$ to $\omega = 2\Delta$ (or broader). Such plot might be valuable and instructive for some infrared spectroscopy measurements on highly inhomogeneous superconductors.

We thank the referee for their questions.

From the experimental point of view, the equivalence between J_s and L_k is valid only in the circuit's frequency regime $\hbar\omega = h\nu \ll \Delta$. From Figure 2 (red curve, right axis), one sees that at optimum doping $\Delta^{\rm opt} \lesssim 25\,\mu$eV (corresponding to $T_c \lesssim 200\,$mK), giving an upper bound for the frequency of about $\nu = 6\,$GHz. We therefore designed a circuit with a maximum resonant frequency of 500 MHz that satisfies the condition $h\nu \ll \Delta$. On the other hand, the bandwidth of microwave components such as the cryogenic amplifier and the directional coupler imposes a low-frequency cut-off around 100MHz, hence our choice to work in the [100MHz-500MHz] range.

Experimentally, assessing the frequency dependence of σ_1 and σ_2 would require to develop a new broadband measurement setup. However, this can be easily investigated numerically by solving the RIN model at different frequencies. In Fig. R1, we show how the real and imaginary part of the complex conductivity $\sigma_1(\omega, T)$ and $\sigma_2(\omega, T)$ vary at different resonance (angular) frequencies in the specific range of the experimental measurements (see also Figure 2 and 3 of Singh Nat. Comm. (2018) 9,407). The set of parameters is exactly the same as the OD and UD calculations presented in the manuscript and hence the red curves for $\omega_0 = 2 \cdot 10^9\,\mathrm{s}^{-1}$ are exactly the same. Besides, our model is valid in stationary conditions, in the regime of frequency presented. At much higher - or even lower - frequencies the RIN model would of course be computationally valid, while other effects should eventually be included to reproduce the physical system, such as non-linearity effects in the reactive (inductive) response. In Venditti et al., Nanomaterials 2021, 11(8), 1888 we studied - in effective medium approximation - the dissipative effects that might arise from thermal excitation of vortices in an even lower frequency regime ($\omega = 0.1 \cdot 10^6\,\mathrm{s}^{-1}$).

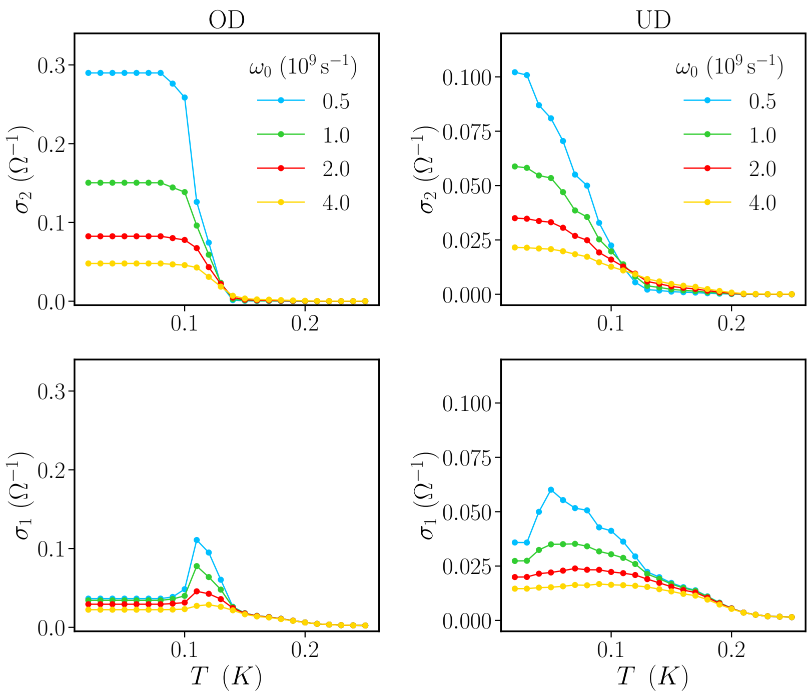

FIG. R1: Note that the red curves are the ones presented in the manuscript.

We have decided not to include this analysis in this publication, which already contains a large amount of information. A discussion of the frequency dependence of the complex conductivity would be indeed a very interesting topic for our scientific community, provided that one takes into account the possible dissipative and/or nonlinear effects that might arise in different ω regime. This would hence require a devoted work, that accounts for experimental evidences.

In the revised version of the manuscript, at the end of Section 3, we better clarified the choice of the frequency ω_0.

(iii) For the readers less familiar with 'random resistor network' (RRN) and its RIN extension it would be very helpful to learn more about this computational algorithm. I urge the authors to expand their section 3 including more details in order to make the paper be self-contained. Requested changes Points (i)-(iii) in the main report.

We thank the referee for raising this point. We actually realized that the RIN computational algorithm was never presented in detail, neither in this work nor in our previous ones. We are thus happy to take their suggestion to expand Appendix A with a specific subsection of computational details.

We thank the referee for their suggestions and hope that they will now consider our work ready for publication in SciPost.