Response to Reviewer #2

We thank the reviewer for their careful reading and positive review of our work.

STRENGTHS

(1) Rigorous investigation of generalized XXZ model by Montecarlo
(2) Thorough explanation of all the criteria and approaches used in the numerical analysis
(3) Interesting final results and general discussion of the phase diagrams

We thank the referee for the appreciation of our work.

WEAKNESSES

(1) The phenomenological nature of the approach, in connection with the Cuprates, should be acknowledged or discussed further
(2) Slightly lengthy in some parts
(3) Clarity of figures might be improved

We thank the referee for the constructive criticism that helped us to improve the manuscript as explained below.

REPORT

The manuscript provides a detailed investigation, by means of classical Montecarlo, of a generalized XXZ (anisotropic Heisenberg) model. Besides in-plane vs out-of-plane anisotropy, this includes a potential barrier separating the competing phases which are studied (order in or out of plane) and a random magnetic field.
The study is very careful and appropriate caution is used (and very well explained) to handle finite-size effects, meta-stability and spinodals, and equilibration. Phase diagram are traced and discussed, and in particular the influence of the random field is addressed.

We thank the referee again for the positive report on our work.

The results are used to discuss the competition of charge-ordering and superconductivity in the cuprates. The parallel between the final phase diagram is apparent, thus I deem this description relevant. However a focus should be put on the phenomenological nature of this analysis.

We agree that the phenomenological character was not emphasized enough and we have now added "phenomenological" to the title and in several parts of the text.

Indeed, the studied model describes emergent classical degrees of freedom, which are linked to the microscopic physics of cuprates through a series of nontrivial assumptions: the attractive Hubbard model is already a qualitative approach to cuprates, and albeit the mapping in a repulsive model is rigorous, the large U limit of the latter is not obviously justified (and this is not done in the text), while the justification for its classical solution and its implications are just touched upon.

We now discuss the large U limit in the context of a phase fluctuation scenario for not too large doping on pag. 6 (second paragraph). The classical solution stems from the fact that the model is expected to be in the "renormalized classical regime". Essentially, as long as the stiffness is large enough (both for superconductivity and charge order), fluctuations involve a large number of pseudospins which move coherently and can be represented with a semiclassical variable. We estimated the number of pseudospins involved using the existing results in the literature. We believe that this issue is now sufficiently explained and we indicated appropriate references for a more in-depth analysis.

Furthermore, there are three parameters in the model which are explored and/or adjusted (reasonably indeed, but their values are not linked to the cuprates in the text).

In general, since these parameters can not be estimated from first principles, they should be obtained by comparing them with the resulting phase diagram. We have now discussed the values of the J parameter in the phase diagram for the 3D system section. We discuss briefly also the disorder parameter.

Finally the phase diagram wanted features of having a sizeable T_{CO} at $\alpha \sim 1$ and a "foot" of superconductivity at low temperatures for $\alpha \gtrsim 1$ are crucially obtained only when including the inter-layer couplings and for a chosen set of values. Again, all these assumptions look reasonable, but the large number of parameters and the specificity of the approach make this a successful phenomenological description (providing an insight into some aspect of the physics) rather than a compelling simplification of the microscopic physics.

I recommend publication but suggest some tuning of the narrative.

We thank the referee for the recommendation. We have now tuned the narrative to clarify the points raised and in particular emphasising the phenomenological aspects.

REQUESTED CHANGES

The aforementioned aspect of ad hoc and phenomenological approach should be better highlighted, from the outset (for example in the abstract and introduction).

In the revised manuscript we have better clarified this aspect.

Not a requested change, rather a suggestion. The article is quite long. Some of the thorough explanations are very welcome indeed, but some parts feel quite slow instead. For instance the B=0.2 diagram of the clean system is practically identical to the known one with B=0. I understand that in this part some concepts used in the subsequent sections are pedagogically introduced, but I wonder if some of this can be moved to the appendix in order to shorten the main narrative.

We appreciate this suggestion and we implemented it by moving part of the discussion of the clean system in appendices. However, there is a substantial difference between the bare XXZ model ($B = 0$) and our clean model with $B = 0.2$, for example, the presence of metastability. There is also a subtle but conceptually relevant difference in the positive slope of the transition line between SC and CO. This is an indication that SC is entropically favoured. Because of that, we maintained part of our results about the clean system in the main text. Nevertheless, we agree that Section 3 was too long and we moved some discussions to the appendix (details of the spinodal and first-order transition points, phase diagram for $B = 2$).

The figures are at times too small (indeed pdf figures can be zoomed in without quality loss, but if they can be made more readable at the page scale it is better, and in a printed version the information can be hardly readable).
Examples:
-in both Fig. 6a and in Fig.11 the points around alpha=1 are hard to visually disentangle. Fig. 11 would benefit simply from being text-wide.
- The snapshot in Fig 10 and the data in the small panels in Figs. 12, 13 and 15 are too small. Even if their main message is well conveyed at this scale, in order to actually see the data one has to zoom in. Where possible their readability should be improved.

Typos:
- pag.12: "bicrtical" should be "bicritical"
- pag.15: Ref. to Sec. 3 should be to Sec. 3.3
- pag 15: Ref. to Fig5c should be to Fig.5f
- pag 20: Ref to "blue" in Fig.11 should be to "purple" Section 5. Reference to a "three-dimensional phase diagram" (both in the text and the title) should rather be to a "phase diagram of a three-dimensional system" or something similar.

We thank the referee for their careful reading of the manuscript and for spotting those corrections and typos. We have addressed all those typos and modified the figures in order to make them more readable.

Validity: High. Significance: High. Originality: High. Clarity: Top. Formatting: Good. Grammar: Excellent.