Reply to Report 2 by Referee 2:

1. **Q:** *What is known about electric conductivity and chemical potentials in PT-symmetric systems?*
A: To the best of our knowledge, not many results exist on electric transport in non-interacting, and particularly even less in interacting PT-symmetric systems. In [40], the authors study the conductivity in a 1-dimensional non-Hermitian Dirac model, which has a PT-symmetric phase and a PT-broken phase at zero temperature. They find that the sum rule holds in both phases, as a consequence of the $U(1)$ gauge invariance. In [83], a PT-symmetric superconductor junction was considered, where the U(1) symmetry is spontaneously broken and the current-voltage characteristic strongly depends on the chosen inner product. For the model in our paper, the $U(1)$ symmetry is explicitly broken by the non-Hermitian source deformations, which constitutes a physically different situation from [40,82,83].
2. **Q:** *How is this affected by weak coupling vs strong coupling?*
A: Other than from our work, to date no results for electric transport in PT symmetric non-Hermitian systems exists in the strongly interacting regime. In our holographic model, both the UV and IR fixed points are conformal. The holographic system is gapless, and also does not have a quasi-particle description. The UV fixed point leads to the limiting behavior $\sigma(\omega) \to 1$ at high frequency. In addition, at zero temperature, the conductivity does not have a hard, but rather a soft gap, which arises in such holographic models [73]. This is to be contrasted to the rotor model, where the pseudo-Goldstone mode α in (69) and the fluctuation (71) are particle excitations. On the other hand, the model of [40] has fermionic quasiparticle excitations, and in the PT-symmetric phase, the fermion has a finite mass and the conductivity has a hard gap at half-filling. These differences between the holographic model and models of quasiparticle transport are qualitatively the same as in Hermitian systems, at least for weak PT deformation in phase I. Furthermore, our results for the conductivity in the strong PT deformation regime in phase II and III are completely novel.
3. **Q:** *How do the results presented here compare with previous expectations? For instance, was it expected that the FGT sum rule should actually break down?*
A: There are several reasons that could lead to an expectation of the sum rule not holding in our holographic model: 1) [56] showed that stability in the vector channel is necessary for the FGT sum rule to hold. On the other hand, [1] already showed that there is an unstable mode in the scalar $(\delta A_t, \delta\phi)$ channel. Based on this, it was not clear to us that no instability should be present in the vector channel either. We performed a nontrivial check by showing that such an instability is absent, and hence the FGT sum rule actually holds. 2) Usually, it is expected that charge conservation leads to the sum rule [40]. There two reasons

to believe that this argument may not hold in our model: First, the instability in the scalar channel involves the charge density mode δA_t, and hence evolving this instability might violate charge conservation. Second, we already argue in the reply to question 8 of Referee 1 that the charge conservation equation in our model does not hold as an operator identity, but only in the static state. Hence, an independent check of the FGT sum rule also seemed necessary from the point of view of charge conservation. Our preliminary conclusion is that the implication of [Phys. Rev. B **71**, 104511 (2005)] might rely on too strong assumptions, and the charge conservation in the considered state (not in all states) may be enough to show the FGT sum rule.

4. **Q:** *What is the relation between the appearance of complex VEVs, metrics and temperatures in phase III of the model as compared to other instances where these features have appeared in holography? For instance, is there a connection with the so-called complex CFTs?*

   A: Complex CFT is one kind of non-unitary CFT. It appears when the Breitenlohner-Freedman bounds for some operators are broken and a pair of complex fixed points appears with the scaling dimensions being complex conjugate pairs [Phys. Rev. D **80**, 125005]. However, in our PT symmetric system, the sources are complexified, while the scaling dimension of the scalar operators stays real in both the UV and IR fixed points, i.e. we are dealing with real CFTs. We hence do not see a direct connection with complex CFTs.