Resubmission letter

Dear editor,

Herewith we resubmit our manuscript titled “Dual symplectic classical circuits: An exactly solvable model of many-body chaos” to SciPost. We would like to express our gratitude to the reviewers for their time and their invaluable feedback on the initial submission. In response to their constructive comments, we have made improvements to our work, enhancing its scientific rigour and clarity. Please see our response below.

Yours sincerely,
Alexios Christopoulos (on behalf of all coauthors)

(We use the Italic font style to indicate the reviewers' comments and reports)

1 Report 1

- *Strengths*
  - *1. Generalizes ideas from dual unitary circuits to classical deterministic ones*
  - *2. Proves several exact results, in particular for two-point correlators*
  - *3. Brings methods more commonly used in quantum dynamics (circuits, tensor networks) to classical many body*
  - *4. Shows explicitly (and in full detail for a class of examples) how deterministic overall dynamics contracts to stochastic dynamics for a subset of degrees of freedom (2-pt correlators in this case)*
- *Weaknesses: None*
- *Report: This is an important paper which adds to the growing corpus of work on "dual" circuit dynamics, an area of much current study started by one of the authors and coworkers. While most work has been on quantum circuits with dual unitary dynamics (i.e. unitary both for time and space*

*propagation) these ideas can also be applied to classical systems. This is what this paper does by considering dual *symplectic* (i.e. phase space volume preserving) classical dynamics. The key property is that of (let's call it) unitality in both space and time, see Eq. (15), whereby time and space propagation leave the flat probability vector invariant (both forward and back). As for DU circuits, several results follow immediately. The focus of interest here is on two-point correlators: they are non-trivial only in rays, and their dynamics reduces to a stochastic one. These general and exact results are fleshed out for class of problems corresponding to coupled dynamics of spherical spins, for which many results are made explicit. One can immediately think of follow up questions to this work, so it is clearly opening up avenues for further research. Except for very minor changes (see below) the paper can be published as is.*

We would like to thank the reviewer for his/her positive comments on our work.

We proceed, by mentioning the details of the changes made according to the requests of the reviewer:

- *Montecarlo → Monte Carlo in the abstract.*

  We corrected this typo on p.1.

- *Please clarify what the "hooks" are meant to be at the right and left edges of Fig.2.*

  We added an explanation in the caption of Fig.2, where we clarified that the "hooks" or the "curly" edges as we denote them represent the periodic boundary conditions of the model.

- *Present the Appendices in the same order as they are mentioned in the main text.*

  We believe that the reviewer is right and we changed the order of the appendices, such that they appear in the same order as in the main text.

2 Report 2

- *Report: In this work, the authors study a special type of classical dynamical system known as dual-symplectic circuit. It consists of a discrete set of classical variables updated using local rules in a discrete fashion, similarly to classical cellular automata. In addition to symplecticity, these dynamics are also symplectic "in the rotated channel". This is a very strong mathematical property that allows for the derivation of analytic results. This fact is remarkable because the model is not integrable in the traditional sense. This set of models, which are natural classical versions of so-called dual-unitary circuits previously studied, were not introduced in* *this paper. However, the authors develop a general formalism to compute analytically dynamical correlation functions, similar to the quantum case. The general formalism is nicely applied to one specific model (the Ising swap model) and the predictions are tested quantitatively against numerical Montecarlo data. Overall, I think this is a strong paper. Although perhaps not extremely innovative, it fills a gap in the literature, by extending to the classical case the formalism previously developed in the context of dual-unitary quantum circuits. The draft is well written and all the predictions are supported by convincing numerics. I don't have any particular comment on how to improve the readability of the draft, so I recommend publication of the manuscript as is.*

We would like to thank the reviewer for his/her positive remarks.

We proceed with answering the two questions of the reviewer:

- *First, I believe that, compared to the quantum case, this model could be useful to study more easily the periodicity of the classical orbits and how this depends on integrability/non-integrability of the model. It would be interesting to study in particular the scaling of the orbit sizes with the system size. Do the authors have some intuition about this aspect?*

  The spectral form factor (SFF) can be a tool that can help study the periodic orbits of the classical model since in the classical limit the SFF at time t is related to the number of periodic orbits of size t. There has already been some research about the SFF of dual unitary circuits ("Random Matrix Spectral Form Factor of Dual-Unitary Quantum Circuits") where SFF was calculated explicitly for a family of dual unitary circuits in the thermodynamic limit, showing agreement with random matrix theory, thus indicating strong chaoticity of these models. Although we have not carried out the classical limit of these models explicitly, it is reasonable to expect that, through the classical limit, these results translate directly into the dual-simplectic models. This suggests that (modulo discrete symmetries such as time reversal), dual-simplectic models have a single periodic orbit for each length t. However, direct even numerical verification of this statement is not easy and remains an interesting prospect for the future that is now mentioned in our conclusion.

- *Second, would it be possible to also generalize, say, tri-unitary quantum circuits (as introduced in "Triunitary quantum circuits") to the classical case?*

  In the paper "Triunitary quantum circuits", 3-site gates are defined and dubbed tri-unitary since they have the property of being unitary in three different arrows in time. Then, the authors work on the "folded picture" in order to extract a diagrammatic proof of their results for the 2-point function. The "folded picture" is diagrammatically equivalent to the one on the density space for a classical system where the dynamics are being performed by the Frobenius–Perron operator of the local gate. We believe that, using the same definitions and methods as in the dual-classical case one can generalise it to a tri-symplectic case, where like in the tri-unitary case we demand our 3-site local gate to be symplectic in three directions of time. Moreover, in the classical case the diagrammatics represent integrals over the phase space and a change of variables is what makes them being interpreted in each of these directions of time where the local gate is being replaced by the symplectic gate that propagates the state in the respective direction of time. In order to ensure equivalence of these diagramatics one has to impose extra conditions which come from the generalization of Eq. (15) (of the resubmitted version) in the three directions of time by demanding that the Jacobian coming from the change of variables is 1.

We proceed to the changes made in the main text of the paper.

- *"Interestingly, dual unitary quantum circuits can exhibit strongly chaotic quantum dynamics, whose classical simulation is in general expected to be exponentially hard in system size".We believe it would be important to cite "Computational power of one- and two-dimensional dual-unitary quantum circuits where a rigorous result along these lines was proven."*

  We agree with the citation suggested and we would like to thank the reviewer for his/her suggestion. We added the respective citation at the point of the main text that, the reviewer mentioned (second paragraph of p.2).

- *Finally, two very trivial comments. It appears T. Prosen is misspelled (I believe?) as "T. c. v. Prosen" in multiple entries in the literature. Also, journal names appear sometimes in short-hand notation (e.g. Phys. Rev. Lett.), sometimes with full names (e.g. Physics Letters A). The authors might want to make the notation uniform.*

  We thank the reviewer for the comment and we apologise for the mistake. We corrected the names in "References" to T. Prosen.

3 Report 3

- *Strengths:*
  - *1- Exactly solvable models of classical dynamics.*
  - *2- Natural generalization of dual-unitarity to classical dynamics.*
  - *3- Detailed analysis and exact results for a representative example.*

- *Report: In this work, the authors introduce the notion of "dual symplectic classical circuits" as the classical analogue of "dual unitary quantum circuits", which were recently introduced as exactly solvable models of quantum chaos in which certain dynamical properties (e.g. correlation functions) can be calculated exactly. These calculations are often done graphically, and the authors here show how demanding symplecticity in the time and space direction results in identical graphical identities, from which the calculations from dual-unitarity can be extended to classical circuits. Specifically, it is shown that in these models two-point correlation functions vanish everywhere except on the edge of the causal light cone, where they can be calculated using a transfer operator formalism. In the specific case of an Ising Swap model with additional single spin rotations, the authors explicitly analyse this transfer operator. It is shown that the transfer operator conserves total angular momentum and various exact results on the eigenspectrum are presented, including autocorrelation functions for the S_z spin component. The results are interesting, onvincing, and well presented. Furthermore, there are various results in the growing literature on dual-unitarity that can subsequently be studied in these classical circuits, such that this work opens up new pathways in this research direction. As such, I am happy to recommend this work for publication in SciPost Physics provided some questions/comments are addressed.*

We are grateful to the reviewer for the positive assessment of our work.

We proceed with the requested changes proposed by the reviewer:

- *One of the most remarkable properties of dual-unitary circuits is that it could be explicitly shown that these circuits satisfy the usual 'definition' of quantum chaos, with major contributions from one of the authors. However, even though the title of this work is "An exactly solvable model of many-body chaos", there is no discussion about whether or not these circuits are chaotic (although the ergodicity is discussed). Since classical chaos is well defined, it would be interesting if the authors could explicitly comment on the chaotic properties of their analysed Ising swap model. Lyapunov exponents are mentioned in the introduction, but then not discussed in the main text. In a related comment, classical and quantum notions of integrability are defined differently, and it would be interesting if the authors could comment on how these notions are satisfied in the presented model when mentioning the integrable points of the Ising swap gate*

  The reviewer is perfectly right and we are thankful for his comment. For the purpose of this, we added above Eq. (28) the paragraph "These models are also known [...] is a characteristic property of chaotic systems.". With this additional text, we explain how one can generate more conserved quantities through the local conserved ones and mention that the Lyapunov spectrum vanishes for an integrable system for which we added the reference [34]. Moreover, we show that the Ising Swap model demonstrates chaotic behaviour away from integrability and for this purpose, we added in Section 4, the plot:

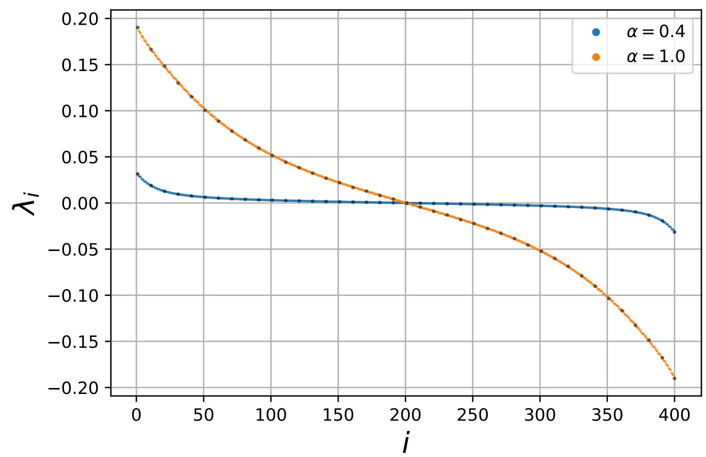

Figure 1:

for two different values of the coupling constant α of the Ising Swap model and for $\beta, \gamma = \sqrt{2}\pi, \sqrt{3}\pi/2$. This plot demonstrates a positive maximal Lyapunov and thus sensitivity to initial conditions which is a characteristic property of chaotic behaviour.

- *On page 3, it is mentioned that the spectrum of the Jacobian needs to include pairs of eigenvalues and that the Lyapunov exponents appear in pairs. Could the authors provide either an argument or a reference as to why that is the case?*

  We agree with the reviewer and thus we add the reference [28], which provides a more detailed proof. In a few words, the proof is based on the relation $Dg^T \omega Dg = \omega$ for the Jacobian Dg of a symplectic map g. Using this relation, one can prove that an eigenvalue g_i of Dg is also an eigenvalue of Dg^{-1} and thus they come in pairs of $g_i, 1/g_i$. Because the Lyapunov exponents are obtained from the logarithm of the magnitude of the eigenvalues g_i of the Jacobian matrix of the map, in the case of a symplectic map $\lambda_i = \log|g_i|, \lambda_{i'} = \log|1/g_i| = -\lambda_i$.

- *In the same section, when introducing the model, the authors write "symplectic maps always involve d-pairs of conjugate variables,the configuration q and the momentum p, which can be seen as the coordinates of a 2d dimensional manifold", before discussing such properties as those in my previous point. However, in the specific example considered in this manuscript the* *authors consider the dynamics of spin variables. It might be worthwhile to also discuss/mention spin variables in that section and the connection with typical pairs of conjugate variables.*

We agree with the suggested change from the reviewer and thus, we added below Eq. (24) an explanation and the exact relation that connects the spin variables to the conjugate ones. In more detail, we mention that there is no unique choice for a set of conjugate variables since a change of coordinates under a symplectic transformation, leads you to another set of conjugate variables, but we focus on the pairs of φ_i, z_i, which are the two out of the three cylindrical coordinates: the azimuthal angle and the cartesian coordinate along the z-axis. Moreover, we present the typical Poisson brackets that conjugate variables like φ_i, z_i satisfy, just above Eq. (25) and the relation between the classical spin variables S_i^x, S_i^y, S_i^z and φ_i, z_i at Eq. (25).

- *Small typo below Eq. (25): "The transfer operator is just the Perron–Frobenius of ..."*

  We corrected the missing word "operator", just below Eq. (28).

- *Conservation of total angular momentum typically follows from rotational symmetry, and it might be useful to discuss this symmetry in the main text. Right now it is mentioned below Eq. (B.6) that "This is not surprising, since as we can see from (B.3),(B.6) the transfer operator is just a composition of rotations, which preserve the total angular momentum." It would be useful to include such a discussion in the main text.*

  We thank the reviewer for his suggestion and we think that a discussion about rotational symmetry in the main text makes the manuscript more complete and coherent. In particular, we added on the paragraph below Eq. (28) ("Rotations preserve [...] linear superposition of rotations"), some extra discussion, where we mention that indeed the transfer matrix $\mathcal{F}$, being somewhat a composition of rotations, commutes with the total angular momentum J^2 but not with arbitrary rotations. Moreover, the local gate $\mathcal{P}_{\Phi_{\alpha,\beta,\gamma}}$ includes a non linear rotation which, is the Ising gate. Because of this non linearity as we prove in Appendix E, $\mathcal{P}_{\Phi_{\alpha,\beta,\gamma}}$ is not block diagonal in the eigenvalues of J^2 and thus it does not commute with neither J^2 nor arbitrary rotations. To conclude, the transfer matrix preserves the total angular momentum but the model in general does not demonstrate rotational symmetry.

We proceed with the list of changes made according to the reports from the reviewers. We also include the changes we made to the manuscript outside the framework of the reviewing process.

4 List of changes

- In throughout the text we replaced "Perron–Frobenius" with "Frobenius-Perron" and "SWAP" with "Swap".
- p.1 Abstract: We replaced "Remarkably, for these models, the rotational symmetry . . . " with "Remarkably, expressing these models in the form of a composition of rotations . . . "
- p.1 Abstract: We replaced Montecarlo with Monte Carlo
- Introduction ,p.2, second paragraph: We added an additional reference, the reference [19]
  "[19] R. Suzuki, K. Mitarai and K. Fujii, Computational power of one- and two-dimensional dual-unitary quantum circuits, Quantum 6, 631 (2022), doi:10.22331/q-2022-01-24-631."
- at the end of section 2: We added a reference, the reference [28]
  "[28] E. Ott, Chaos in Dynamical Systems, pp. 251–258, Cambridge University Press, 2 edn., doi:10.1017/CBO9780511803260 (2002)."
- Section 3.1, at the caption of Fig.2 : We added the following text " with the "curly" edges indicating the periodic boundary conditions of the model." at the caption of Fig. 2.
- Section 3.2, the second half of the first paragraph, above Eq. (15): We added the following:

  "The dual picture allows for diagrams like the one in Fig 2, to be interpreted in the space direction too from left to right, with the exchange of $\Phi \to \tilde{\Phi}$ or even from the right to the left where the dual map is defined as in Fig. 3 but, with the exchange of the legs of Φ along the other diagonal. However, these diagrams are just graphical representations of integrals over the phase space M_N and the passing to the dual picture is a change of integration variables, which leads to a factor coming from the Jacobian of the transformation. In order for both pictures to be equivalent under this change of variables and not carry these type of Jacobian factors, one should impose that this Jacobian is 1 for both of the left to right and right to left directions in space and thus, the local gate should satisfy the following conditions:

  $$\left|\det\left(\frac{\partial\Phi^1(\vec{X}_1,\vec{X}_2)}{\partial\vec{X}_2}\right)\right| = \left|\det\left(\frac{\partial\Phi^2(\vec{X}_1,\vec{X}_2)}{\partial\vec{X}_1}\right)\right| = 1 \quad ,\forall\vec{X}_1,\vec{X}_2 \in M\times M \tag{15}$$

  where $\Phi^{1,2}$ are the single site outputs of the local gate defined as $\left(\Phi^1(\vec{X}_1,\vec{X}_2),\Phi^2(\vec{X}_1,\vec{X}_2)\right) = \Phi(\vec{X}_1,\vec{X}_2)$. We provide an explicit proof of (15) in Appendix A. In addition, by definition the dual map is an involution and so the dual of the dual picture should be the original one with Φ. In order to, assure that the change from the original picture in the time direction to the the dual one and vice versa is equivalent, then (15) should respectively hold for the dual map. We actually, prove in Appendix A, that the condition (15) for Φ is enough for this to be true."

- Section 3.2, above Eq. (16): We added the following part
"that also satisfies (15). We stress ..."
and "We stress that Eq. (15) is actually crucial and follows naturally in dual simplectic circuits which are obtained through a limiting procedure of dual-unitary quantum circuits with a finite a discrete local hilbert space. In fact, there has already been some research on dual symplectic circuits where Eq. (15) does not hold; in particular, in integrable circuits with non-abelian symmetries, it has been demonstrated [15] that 2-point dynamical correlations follow Kardar–Parisi–Zhang (KPZ) universality and are not restricted to the edges of the light cone, in contrast with what we prove here for dual-simplectic circuits where Eq. (15) holds"

- Section 4, above Eq. (22): We added Table 1, which includes a table with all the different levels of ergodicity that our model can demonstrate. In particular, we refer to the following table:

eigenvalues $\mu_i, \quad i \neq 0$	*non−interacting*	*non−ergodic*	*ergodic, non−mixing*	*ergodic, mixing*
$\mu_i = 1$		$\mu_1, \mu_2, \ldots, \mu_{j<\infty}$		
$\lvert\mu_i\rvert = 1$	$\mu_1, \mu_2, \ldots, \mu_\infty$	$\mu_{j+1}, \mu_{j+2}, \ldots, \mu_{j+m<\infty}$	$\mu_1, \mu_2, \ldots, \mu_{j<\infty}$	
$\lvert\mu_i\rvert < 1$		$\mu_{j+m+1}, \mu_{j+m+2}, \ldots, \mu_\infty$	$\mu_{j+1}, \mu_{j+2}, \ldots, \mu_\infty$	$\mu_1, \mu_2, \ldots, \mu_\infty$

- Section 4, below Eq. (22): We replaced the symbol for the axis α with n.

- Section 4, above Eq. (28) we added the figure Fig. '1 as Fig. 5 along with the caption:
"The Lyapunov spectrum λ_i of the Ising Swap model, for two different coupling constants $\alpha = 0.4, 1$, angles $\beta = \sqrt{2}\pi, \gamma = \sqrt{3}\pi/2$ and for system size $N = 200$. It was created for $t = 800$ and a sample size of $N_{sample} = 10^4$ initial states drawn from the uniform measure. The black circles represent the Lyapunov spectrum at every 10 exponents, at time $t = 700$, showing an excellent time convergence for λ_i. The spectrum is symmetric with respect to the horizontal axis, as expected for a symplectic system and has a positive maximal Lyapunov exponent, indicating chaoticity for $\Phi_{\alpha,\beta,\gamma}$."

- Section 4,below the Eq. (24) of the manuscript: We added the following:
" The spin variables as we can see from (24), are not the pairs q, p of conjugate variables that we expect in symplectic dynamics. In general, there is not a unique choice of conjugate variables, since a symplectic transformation maps you from a set of conjugate variables to another. However,

here we choose the pairs φ_i, z_i with z_i being the cartesian coordinate along the z-axis and φ_i the azimuthal angle of the i-th site and so they satisfy:

$$\{\varphi_i, z_j\} = \delta_{ij} \quad , \quad \{\varphi_i, \varphi_j\} = \{z_i, z_j\} = 0 \tag{25}$$

The spin variables are just vectors of the unit sphere meaning that they are related to φ_i, z_i as:

$$S_i^x = \sqrt{1 - z_i^2}\cos(\varphi_i) \quad , \quad S_i^y = \sqrt{1 - z_i^2}\sin(\varphi_i) \quad , \quad S_i^z = z_i \tag{26}$$

and one can easily check that (26) satisfies the SO(3) Poisson bracket (24). ”

- Section 4, just above Eq. (27) : We replaced ”It is easy to verify that the space-time dual of the gate (21), as defined in Fig. 3, has a similar form”with
”We explicitly demonstrate in Appendix C that, (22) satisfies (15), allowing us for equivalent interpretations of the diagrams in both the time and space direction. Following the same method as employed in Appendix A, one finds that, the space-time dual of our model is defined as:”

- Section 4 , above Eq. (28) paragraph on the top: We added the reference [34]
”[34] H.-D. Meyer, Theory of the liapunov exponents of hamiltonian systems and a numer- ical study on the transition from regular to irregular classical motion, The Journal of Chemical Physics 84(6), 3147 (1986), doi:10.1063/1.450296.”

- Section 4 , above Eq. (28) : We added the following text:
”Later, in (33) we provide analytical results for the auto-correlation of the z-components at the integrable points, when they do not decay to zero. In general, every scalar that depends on the sum of the z-components along the aforementioned bipartitions will be conserved. At the integrable points of our parameter space, trajectories in phase space are bounded on invariant tori and the Lyapunov spectrum vanishes [34], whereas away from those points, chaotic behaviour is expected to arise. In Fig. 5 we present some examples of the Lyapunov spectrum at chaotic points of our Ising Swap model, where it demonstrates a positive maximal Lyapunov exponent and thus sensitivity to initial conditions, which is a characteristic property of chaotic systems.”

- Section 4, just below Eq. (28): We replaced ”The transfer operator is just the Perron–Frobenius of . . . ” with ”The transfer operator is the Frobenius-Perron operator of. . . ”

- Section 4, paragraph below Eq. (28): We added the text:
”Rotations preserve the total angular momentum and since, $\mathcal{F}$ according to (28) is a composition of rotations it shares the same property, as proven

in Appendix D. More explicitly, we denote as $J_i, i = x, y, z$ the generators of single site rotations and $J^2 = \sum_i J_i$ as the angular-momentum-squared which satisfies $[J^2, J_i] = 0, \forall i$ and thus, commutes with every rotation operation. Then, indeed $\mathcal{F}$ commutes with the angular momentum and thus has block diagonal form in its eigenvalues, as we demonstrate in Appendix D. However, this is not a consequence of an underlying rotational symmetry but rather of the specific form of the local gate $\mathcal{P}_{\Phi_{\alpha,\beta,\gamma}}$. Indeed, the Ising swap gate in $\mathcal{P}_{\Phi_{\alpha,\beta,\gamma}}$ involves a non-linear rotation, i.e. a rotation whose angle depends on the z component of the neighbouring. Because of this non-linearity, it is not block diagonal with respect the eigenvalues of J^2 as we prove in Appendix E. Nonetheless, in going from the local gate $\mathcal{P}_{\Phi_{\alpha,\beta,\gamma}}$ to the transfer operator $\mathcal{F}$, the neighbouring site is, by definition Fig. 4, in the equilibrium state, so that its z component can be integrated over, thus leading to the operator $Q(\alpha)$, which is a linear superposition of rotations"

- Section 4, paragraph above Eq. (29): We replaced "We choose the coordinates z, φ being respectively the z cartesian component and the azimuthal angle." with "We choose the conjugate variables z, φ for the parametrization of $\mathcal{S}^2$." We also replaced "As we rigorously prove in Appendix B, the transfer operator $\mathcal{F}\ldots$" with "As we already mentioned, the transfer operator $\mathcal{F}$ ..."

- Section Conclusion: We replaced

  "It is important to mention that our method is valid not only for dual-symplectic systems. Specifically, it is easy to check that any local gate Φ which is volume preserving that also has a volume-preserving dual map $\tilde{\Phi}$, satisfy (15) and thus exhibit the same diagrammatic behaviour. "

  with
  "We would like to stress, that our method is valid not only for dual-symplectic systems, as it is easy to check that any local gate Φ which is volume preserving and which also has a volume-preserving dual map $\tilde{\Phi}$ and satisfies (15), satisfies also (16) and exhibits the same diagrammatic behaviour."

- Section appendices: We arranged the appendices in the same order as they appear in the main text.

- Section appendices: We added two more appendices A,C under the respective titles: "Dual picture and change of variables", "Diagrammatic equivalence's conditions for $\Phi_{\alpha,\beta,\gamma}$"

- Appendix E, Eq. (E.5),Eq. (E.6),Eq. (E.7): We changed $T(.)$ to $\mathcal{P}_{T(.)}$ and $\Phi_{\alpha,\beta,\gamma}$ to $\mathcal{P}_{\Phi_{\alpha,\beta,\gamma}}$

- We added to Acknowledgement: "We acknowledge fruitful discussions with B. Bertini and Ž. Krajnik which, in particular, helped us to understand the role of the Jacobian when transforming between the time and space directions."