RESPONSE TO ANONYMOUS REPORT 1

We thank the referee for the careful reading of the manuscript and the valuable comments. Please find our responses to the comments and a summary of changes below.

1) I do not understand the first equality in (8). Suppose the space is $\mathbb{R}_t \times M^3$, and j is a one-form. Denoting with i the M^3 directions, and taking $\sqrt{g_3}$ the square root of the metric determinant, then, naively,

$$\int_{M^3} \mathcal{L}_{e^\mu}(\star j) = \int_{M^3} d^3x \mathcal{L}_{e^\mu}(\sqrt{g_3} j^t) = \int_{M^3} d^3x \left(e^\mu \partial_\mu(\sqrt{g_3} j^t) - \nabla_\alpha e^\mu (\sqrt{g_3} j^\alpha)\right) \tag{1}$$

If $e^\mu = \delta_i^\mu$ in general $\nabla_\alpha e^\mu = \delta_i^\mu \Gamma^i_{\alpha i} \neq 0$ if M^3 is curved. I have a similar issue with the equality in (111).

In the present work we are interested only in flat manifolds with global Minkowski coordinates, on which Eq. (8) is valid. We have added a new Footnote after Eq. (8) to clarify this point.

Let us now elaborate on the extension of Eq. (8) in our manuscript to curved manifolds, and for full generality let us consider a continuous p-form symmetry in $d+1$ spacetime dimensions. The conserved current is a $(p+1)$-form such that $\star j$ is a $(d-p)$-form. Therefore the Lie derivative acts as

$$\begin{aligned}\mathcal{L}_e(\star j) &= \mathcal{L}_e\left[\frac{1}{(d-p)!(p+1)!}\sqrt{|g|}\epsilon_{\nu_1\ldots\nu_{p+1}\mu_1\ldots\mu_{d-p}} j^{\nu_1\ldots\nu_{p+1}} \mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d-p}}\right] \\ &= \left[\frac{\epsilon_{\nu_1\ldots\nu_{p+1}\mu_1\ldots\mu_{d-p}}}{(d-p)!(p+1)!} e^\rho\partial_\rho\left(\sqrt{|g|}j^{\nu_1\ldots\nu_{p+1}}\right) + \sum_{r=1}^{d-p}\frac{\epsilon_{\nu_1\ldots\nu_{p+1}\mu_1\ldots\rho\ldots\mu_{d-p}}}{(d-p)!(p+1)!}\sqrt{|g|}j^{\nu_1\ldots\nu_{p+1}}\partial_{\mu_r}e^\rho\right]\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d-p}},\end{aligned} \tag{2}$$

where $\epsilon_{\lambda_1\ldots\lambda_{d+1}}$ is the totally skew-symmetric Levi–Civita symbol with $\epsilon_{0\ldots d} = +1$, and in the second line we have used the standard expression for the Lie derivative of a rank-$(d-p)$ covariant tensor. For each term in the sum, the lower index ρ is understood to replace μ_r.

We can rewrite this as the Lie derivative of a contravariant object by using

$$\epsilon_{[\lambda_1\ldots\lambda_{d+1}}\partial_{\rho]}e^\rho = 0, \tag{3}$$

which is simply the statement that in $d+1$ spacetime dimensions, any object with more than $d+1$ skew-symmetric indices must vanish, since at least one index is guaranteed to be repeated. Applying this to the definition of $\mathcal{L}_e(\star j)$, we obtain

$$\begin{aligned}\mathcal{L}_e(\star j) &= \left[\frac{\epsilon_{\nu_1\ldots\nu_{p+1}\mu_1\ldots\mu_{d-p}}}{(d-p)!(p+1)!}e^\rho\partial_\rho\left(\sqrt{|g|}j^{\nu_1\ldots\nu_{p+1}}\right) + \frac{\epsilon_{\nu_1\ldots\nu_{p+1}\mu_1\ldots\mu_{d-p}}}{(d-p)!(p+1)!}\sqrt{|g|}j^{\nu_1\ldots\nu_{p+1}}\partial_\rho e^\rho\right. \\ &\qquad \left. - \sum_{s=1}^{p+1}\frac{\epsilon_{\nu_1\ldots\rho\ldots\nu_{p+1}\mu_1\ldots\mu_{d-p}}}{(d-p)!(p+1)!}\sqrt{|g|}j^{\nu_1\ldots\nu_{p+1}}\partial_{\nu_s}e^\rho\right]\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d-p}} \\ &= \frac{\epsilon_{\nu_1\ldots\nu_{p+1}\mu_1\ldots\mu_{d-p}}}{(d-p)!(p+1)!}\left[e^\rho\partial_\rho\left(\sqrt{|g|}j^{\nu_1\ldots\nu_{p+1}}\right) + \sqrt{|g|}j^{\nu_1\ldots\nu_{p+1}}\partial_\rho e^\rho - \sum_{s=1}^{p+1}\sqrt{|g|}j^{\nu_1\ldots\rho\ldots\nu_{p+1}}\partial_\rho e^{\nu_s}\right]\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d-p}}.\end{aligned} \tag{4}$$

To obtain the second equality, we have simply swapped the labelling of the dummy indices ρ and ν_s. Note that in the last line, the expression in square brackets corresponds to the Lie derivative of the contravariant *tensor density* with components $\sqrt{|g|}j^{\nu_1\ldots\nu_{p+1}}$, with the second term proportional to $\partial_\rho e^\rho$ giving the non-tensor part of the transformation.

Specializing to the case of $d=3$, $p=0$ and a spacetime $\mathbb{R}_t\times M^3$, we find

$$\begin{aligned}\int_{M^3}\mathcal{L}_e(\star j) &= \int_{M^3}\frac{1}{3!}\epsilon_{\nu\mu_1\mu_2\mu_3}\left[e^\rho\partial_\rho\left(\sqrt{|g|}j^\nu\right) + \sqrt{|g|}j^\nu\partial_\rho e^\rho - \sqrt{|g|}j^\rho\partial_\rho e^\nu\right]\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\mathrm{d}x^{\mu_3} \\ &= \int_{M^3}\mathrm{d}^3x\left[e^\rho\partial_\rho\left(\sqrt{|g|}j^t\right) + \sqrt{|g|}j^t\partial_\rho e^\rho - \sqrt{|g|}j^\rho\partial_\rho e^t\right].\end{aligned} \tag{5}$$

When our space is $\mathbb{R}^3$, with $e^\rho = \delta^\rho_\mu$ on the global coordinate patch, the second and third terms vanish and we recover the first equality of Eq. (8).

To address the referee's concern, we would like to understand what happens on more general curved manifolds. Until now we have not introduced a connection ∇, since the Lie derivative is defined without reference to any metric structure. Nevertheless, after some brief algebra it is straightforward to express (4) in a manifestly covariant manner as

$$\mathcal{L}_e(\star j) = \frac{\sqrt{|g|}\epsilon_{\nu_1\ldots\nu_{p+1}\mu_1\ldots\mu_{d-p}}}{(d-p)!(p+1)!}\left[e^\rho\nabla_\rho j^{\nu_1\ldots\nu_{p+1}} + j^{\nu_1\ldots\nu_{p+1}}\nabla_\rho e^\rho - \sum_{s=1}^{p+1} j^{\nu_1\ldots\rho\ldots\nu_{p+1}}\nabla_\rho e^{\nu_s}\right]\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d-p}}. \quad (6)$$

From this we can read off the general statement: for a *covariantly constant* vector field e^μ, i.e. satisfying $\nabla_\rho e^\mu = 0$,

$$\int_{M_{d-p}}\mathcal{L}_e(\star j) = \int_{M_{d-p}} e^\mu\nabla_\mu\star j. \quad (7)$$

We briefly comment on this generalization in our newly added footnote after Eq. (8) in the manuscript.

2) This is maybe a typo, in (56) one is taking variations with respecto to da and dA, but the formulas suggests the variations are respect to a and A.

We thank the referee for pointing this out, it was a typo and we have corrected it.

3) I'm not sure I agree with the comment below (50) relative to changing coefficients of time and space derivatives, it is not clear how one would maintain gauge invariance, at least with all the formulas that are provided in the language of differential forms.

What we mean by the comment is as follows. Note that the components of the field strengths, $f = \frac{1}{2}f_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$ and $\mathcal{F}_i = \frac{1}{2}(\mathcal{F}_i)_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$ are gauge-invariant. We can combine these components to write down the Lagrangian, depending on the spacetime symmetry of the system. For example, if we introduce an electric and magnetic field for f by

$$f = \widetilde{e}_i\mathrm{d}x^i\wedge\mathrm{d}x^0 + \frac{1}{2}\epsilon_{ijk}\widetilde{b}^k\mathrm{d}x^i\wedge\mathrm{d}x^j, \quad (8)$$

and we can write down a Lorentz-invariant Lagrangian as

$$\frac{1}{2v^2}\left(\widetilde{e}^i\widetilde{e}_i - \widetilde{b}^i\widetilde{b}_i\right). \quad (9)$$

When there is no symmetry between the time and space, the coefficients of the first and second terms can be different as

$$\frac{1}{2(v_1)^2}\widetilde{e}^i\widetilde{e}_i - \frac{1}{2(v_2)^2}\widetilde{b}^i\widetilde{b}_i. \quad (10)$$

4) I'm not sure about the conditions in (60) and (179) and how are they derived from the equations of motion (52)-(55) and (175)-(178), for instance, imposing $f = 0$ in (54) does not give an additional constraint? Same for (177) when $f_i = 0$.

Let us start with the scalar theory. Eq. (61) (previously Eq. (60)) can be derived from Eq. (54) (previously Eq. (53)) as follows. Multiplying through by e^2 and defining $m^2 := (e_1)^2/e^2$, (54) becomes

$$\frac{1}{m^2}d\star\mathcal{F}_i + dx^i\wedge\star f = 0. \quad (11)$$

Taking the limit $m^2\to\infty$, we obtain

$$dx^i\wedge\star f = 0. \quad (12)$$

For any specific component i, this would miss the components of $\star f$ parallel to dx^i. However, since i runs from 1 to d and $\star f$ has at most $(d-1)$ spatial legs, all independent components of $\star f$ are recovered, in which case this is equivalent to

$$\star f = 0. \quad (13)$$

Acting again with the Hodge star, we obtain $f = 0$, Eq. (61).

Turning now to the vector theory, by taking an analogous limit of Eq. (185), we obtain

$$\epsilon_{ijk} dx_j \wedge \star f_k = 0. \tag{14}$$

However, this is in fact *not* equivalent to $f_k = 0$, and we thank the referee for bringing this to our attention. While the result of the calculation is unchanged, one step must be corrected, as we will now explain in some detail.

Expanding the above expression in components, using the fact that $\star f$ is a 2-form in $3+1$ dimensions, we may write

$$0 = \epsilon_{ijk} (\star f^k)_{l0} dx^j \wedge dx^l \wedge dx^0 + \frac{1}{2} \epsilon_{ijk} (\star f^k)_{lm} dx^j \wedge dx^l \wedge dx^m. \tag{15}$$

We can extract components systematically by taking wedge products with dx^μ, and then using the skew-symmetry of the wedge product and the fact that $dx^j \wedge dx^l \wedge dx^m \wedge dx^0 = \epsilon^{jlm0} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0$. Employing the convention that $\epsilon_{0123} = 1$, we also have $\epsilon^{jlm0} = -\epsilon_{jlm0} = \epsilon_{0jlm} = \epsilon^{jlm}$. Then for $\mu = m$, we obtain

$$0 = \epsilon_{ijk} \epsilon^{mjl} (\star f^k)_{l0} = (\delta^m_i \delta^l_k - \delta^m_k \delta^l_i)(\star f^k)_{l0} = \delta^m_i (\star f^k)_{k0} - (\star f^m)_{i0}. \tag{16}$$

Contracting with δ^i_m, we obtain $(\star f^k)_{k0} = 0$, which in turn implies $(\star f^k)_{l0} = 0$ from the above expression. Meanwhile, for $\mu = 0$ we obtain

$$0 = \frac{1}{2} \epsilon_{ijk} \epsilon^{jlm} (\star f^k)_{lm} = \frac{1}{2} (\delta^m_i \delta^l_k - \delta^m_k \delta^l_i)(\star f^k)_{lm} = \frac{1}{2} ((\star f^l)_{li} - (\star f^m)_{im}) = (\star f^k)_{ki}. \tag{17}$$

Finally, let us write these equations in terms of component fields. Explicitly, we have

$$f_i = [(e_i)_k - \epsilon_{ijk} (\mathcal{A}_j)_0] \, \mathrm{d}x^0 \wedge \mathrm{d}x^k + \left[\partial_{[j} (a_i)_{k]} - \epsilon_{il[j} (\mathcal{A}_l)_{k]}\right] \mathrm{d}x^j \wedge \mathrm{d}x^k, \tag{18}$$

where $(e_i)_k \coloneqq \partial_k (a_i)_0 - \partial_0 (a_i)_k$. Meanwhile, the (spacetime) Hodge dual is given by

$$\star f_i = \frac{1}{4} \epsilon^{\mu\nu}{}_{\rho\sigma} (f_i)_{\mu\nu} dx^\rho \wedge dx^\sigma = \frac{1}{2} \epsilon_{klm} (f_i)_{k0} dx^l \wedge dx^m + \frac{1}{2} \epsilon_{klm} (f_i)_{km} dx^l \wedge dx^0. \tag{19}$$

Matching to the above constraints, we find

$$0 = (\star f_i)_{k0} = \epsilon_{jkl} (f_i)_{jl}, \qquad 0 = (\star f^i)_{ki} = \epsilon_{klm} (f_k)_{m0}. \tag{20}$$

Comparing with the explicit components of f_i above, we obtain Eq. (191) and Eq. (192). Thus we have corrected Eq. (191) to depend only on the skew-symmetric part of $(e_i)_k$, which is consistent with the final result that only the symmetric part appears in the low-energy Lagrangian, Eq. (200). We have corrected Eq. (191) in the manuscript, and we have modified the discussion prior to Eq. (191) to reflect the above argument. We have also added footnote 21 following Eq. (192) to clarify our conventions.

An analogous situation arises in the case of general d, for which we have replaced Eq. (249), fixed Eqs. (250) and (269), and modified the surrounding arguments accordingly. We again thank the referee for their helpful comment, which allowed us to fix this subtle yet important point.

5) Can the authors show that the equations of motion obtained from the Lagrangians (67) and (194) coincide with the equations (52)-(55) and (175)-(178)? Given the gauge fixings and change of variables involved it is not immediately obvious.

At the end of Sections III.C and IV.B, we have presented the low-energy effective equations of motion, and explained their derivation from the full equations of motion, for the scalar charge gauge theory and vector charge gauge theory, respectively. In each case, the equations of motion from the low-energy effective Lagrangian can indeed be obtained from the low-energy limit of the full equations of motion, providing a useful consistency check of our results.

SUMMARY OF CHANGES

- We added a new appendix titled "Coupling to complex scalar fields" as Appendix A in which we discussed the coupling of gauge fields of non-uniform symmetries to a theory with a complex scalar field.
- We added a new footnote on page 4 in which we comment on the case of curved spacetime.
- We have fixed a typo in Eq. (56) of the previous manuscript (Eq. (57) in the updated one).
- We have added a new comment in Summary and Discussions (the second point in the list of possible future directions on page 37).
- We have replaced the phrasing "has non-vanishing commutation relations with translations," with "can be written as a commutator of a translation and another charge" on page 6 and page 22.
- We have Footnote 3 on page 6 with the following content:

  "We will use the same symbol Q to denote the charge of a 0-form symmetry and the charge of the corresponding 1-form symmetry that appears as a result of the gauging of the former, to emphasize the connection between these two symmetries. When we wish to highlight the degree of the symmetry, we explicitly write the dependence on the underlying manifold over which the charge density is integrated, e.g. $Q(V)$ and $Q(S)$, where V and S are a d-cycle and a $(d-1)$-cycle, respectively."
- We have corrected Eq. (191), and we have modified the discussion prior to Eq. (191). We have also added footnote 21 following Eq. (192) to clarify our conventions.
- We have replaced Eq. (249), fixed Eqs. (250) and (269), and modified the surrounding arguments.
- We added a derivation of the equations of motion of the scalar charge gauge theory from the equations of motion (53)-(56) in page 12 (see around Eq. (69)-(76)).