RESPONSE TO ANONYMOUS REPORT 2

We thank the referee for the careful reading of the manuscript and the valuable comments. Please find our responses to the comments and a summary of changes below.

1. The use of the same symbol Q to denote both the charge of the original global symmetry and the 1-form symmetry resulting from gauging is very confusing. Not only are these genuinely different symmetries in different theories that act on different operators, but they may even have different symmetry groups. I highly recommend distinguishing between the two charges in some way, perhaps by calling the electric 1-form symmetry charges Q^e, similar to how the magnetic 1-form symmetry charges are denoted by Q^m.

Thanks for the suggestion. Indeed, the charges of 0-form and 1-form symmetries act on different operators, and one of the points we would like to emphasize is that, in gauge theories with fractons (such as the scalar-charge gauge theory), the fractonic property is characterized as a property of a line operator, which is a charged object of a 1-form symmetry. At the same time, a Wilson line operator can be regarded as a worldline of a charged particle, and this allows us to interpret the line operators as "fractons," which suggests a particle interpretation. Indeed, 0-form and 1-form dipole symmetries are not unrelated. For example, the symmetry group ($U(1)$ or $\mathbb{R}$) should be taken to be consistent (see the response to Point 2 below on this). Another relation is that a 0-form nonuniform symmetry and the 1-form nonuniform symmetry that appears as a result of gauging of the former satisfy the same algebra with translations.

Due to these connections, we prefer to use the same symbols for the charges of 0-form and 1-form symmetries. In most cases, we believe they are distinguishable from the context. When we emphasize whether they are 0-form or 1-form charges, we explicitly denote the underlying manifold such as $Q(V)$ and $Q(S)$, where V and S are a $d-$cycle and a $(d-1)$-cycle, respectively, suggesting a "volume" and a "surface" in the case of $d = 3$. We have added explanations on this point in Footnote 3 on page 6. We hope this clarifies the concern.

2. The discussion regarding the normalization of the dipole gauge field (on page 10) seems overly simplistic, and it is not clear to me that the normalization condition of a is independent of the dipole gauge transformations. While the given computation showing why this is the case for the sphere is accurate (eq. 48), it seems like the same argument would fail on a torus that wraps around a non-trivial cycle (as $\int_{S_1} n_i dx^i \neq 0$ for the non-trivial cycle.) In general one would expect that the dipole symmetry could not be $U(1)$ just from the dipole transformations, and this squares with recent discussions of the various continuum limits of a dipole symmetry in [2201.10589]. It would be useful if the authors clarified why they can take the dipole symmetry to be $U(1)$ rather than $\mathbb{R}$.

In our work we consider Minkowski spacetime $\mathbb{R}^{d+1}$ only. In the absence of sources, there are no non-trivial cycles and any spatial submanifold is contractible, so $\int_{S^1} n_i \mathrm{d}x^i = 0$. If a monopole is present and enclosed by our surface, due to the trivial background spacetime we may deform any such surface to a sphere S^2 — for example, in the case of a torus, we may pinch off the torus without introducing any physical singularity. Thus, our calculation for the case of S^2 applies, Eq. (49) holds, and the quantization conditions are gauge invariant. However, we agree that the argument would not always apply on a compact space with non-trivial cycles.

For the case of a $U(1)$ dipole symmetry, even on a non-compact space some exotic features would arise. In particular, the symmetry is consistent and well-defined only at positions x_i for which x_i/l is an integer (l is the length scale introduced in the quantization conditions). For example, in the theory of lattice defects, in which fractons and lineons correspond to disclinations and dislocations, respectively, the parameter l corresponds to the lattice spacing. In essence, when the symmetry is $U(1)$ the spatial "geometry" becomes $\mathbb{Z}$; conversely, in [2201.10589] the authors consider toroidal compactification, such that the spatial manifold in a single direction is $U(1) \simeq S^1$ and the dipole symmetry is $\mathbb{Z}$.

3. The authors don't discuss these gauge theories in the presence of charged matter apart from the Higgs phase. In particular the coupling of dipole invariant matter to tensor gauge fields was systematically constructed in [1807.11479], but it is not clear how the additional gauge fields of the non-uniform dipole gauge theory fit into this construction, or if there is a universal gauge principle for coupling these gauge theories to charged matter. I think it would be useful to discuss how to gauge the dipole symmetry of simple dipole invariant scalar theories in the spirit of [1807.11479], and present the resulting theory of matter coupled to the complete dipole gauge theory.

We thank the referee for the suggestion. In the current draft, we have only assumed that the system has a nonuniform symmetry and that the gauging procedure is applicable to any such model with the corresponding nonuniform symmetry. To facilitate the comparison with former works, following the suggestion, we have discussed a model of a complex scalar field with dipole symmetry and its coupling to gauge fields in Appendix A, which is newly added.

4. On page 6 the authors state that "q_z has non-vanishing commutation relations with translations," which directly contradicts equation (20). I think the statement they are trying to make is that q_z is the commutator of a different conserved charge with translations, which results in particles charged under q_z to be immobile in certain directions. Similar imprecise statements appear in the vector charge gauge theory section.

We thank the referee for pointing this out, we have corrected the phrasing.

5. The authors discuss t' Hooft anomalies of the 1-form symmetries. However there is also a possibility of t' Hooft (or mixed) anomalies between the 0-form symmetries that obstructs the gauging of the non-uniform symmetry in the first place, which the authors do not discuss at all. It would be interesting if the authors had anything to say about the possibility of such anomalies appearing, and what their consequences may be.

We thank the referee for the suggestion. This indeed is a very interesting question and we believe it is worth considering as an independent future project. We have mentioned this in the updated manuscript (the second point in the list of possible future directions on page 37).

SUMMARY OF CHANGES

- We added a new appendix titled "Coupling to complex scalar fields" as Appendix A in which we discussed the coupling of gauge fields of non-uniform symmetries to a theory with a complex scalar field.
- We added a new footnote on page 4 in which we comment on the case of curved spacetime.
- We have fixed a typo in Eq. (56) of the previous manuscript (Eq. (57) in the updated one).
- We have added a new comment in Summary and Discussions (the second point in the list of possible future directions on page 37).
- We have replaced the phrasing "has non-vanishing commutation relations with translations," with "can be written as a commutator of a translation and another charge" on page 6 and page 22.
- We have Footnote 3 on page 6 with the following content:

  "We will use the same symbol Q to denote the charge of a 0-form symmetry and the charge of the corresponding 1-form symmetry that appears as a result of the gauging of the former, to emphasize the connection between these two symmetries. When we wish to highlight the degree of the symmetry, we explicitly write the dependence on the underlying manifold over which the charge density is integrated, e.g. $Q(V)$ and $Q(S)$, where V and S are a d-cycle and a $(d-1)$-cycle, respectively."
- We have corrected Eq. (191), and we have modified the discussion prior to Eq. (191). We have also added footnote 21 following Eq. (192) to clarify our conventions.
- We have replaced Eq. (249), fixed Eqs. (250) and (269), and modified the surrounding arguments.
- We added a derivation of the equations of motion of the scalar charge gauge theory from the equations of motion (53)-(56) in page 12 (see around Eq. (69)-(76)).