’t Hooft Perimeter Law from Ising Mapping

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We can check the perimeter law for the ’t Hooft loop by using the mapping to the Ising model, although we will see that the calculation is very similar. As described in Appendix B, we can map expectation values in a ground state of the deformed toric code model to thermal expectation values in the classical Ising model. This is because the undeformed toric code ground state is an equal sum over flat configurations, which can be mapped to the configurations of the Ising model in the z-basis. The Ising spins reside on the vertices of the lattice and are related to the toric code spins on the edges by $\sigma_i^z = \sigma_{s(i)}^z \sigma_{t(i)}^z$, where $s(i)$ and $t(i)$ are the vertices at the start and end of edge i respectively. For the rest of this discussion Pauli operators indexed by vertices correspond to the Ising basis, while Pauli operators indexed by edges correspond to the toric code basis. This establishes a mapping from flat toric code configurations to Ising configurations. Then the deformation factor $S(\beta)^2$ can be treated as the Boltzmann weight for each configuration and taking the expectation value for certain operators results in a sum over configurations. That is, for an observable $\hat{O}$ which is made of some combination of toric code σ_i^z operators, we have

$$\begin{aligned}\langle \hat{O} \rangle_{\text{Deformed Toric Code}} &= \frac{\langle GS(0)| S(\beta)\hat{O}S(\beta) |GS(0)\rangle}{\langle GS(0)| S(\beta)^2 |GS(0)\rangle} \\ &= \frac{\langle GS(0)| \hat{O}S(\beta)^2 |GS(0)\rangle}{\langle GS(0)| S(\beta)^2 |GS(0)\rangle} \\ &= \frac{\sum_{\{\theta_v\}} O_{\{\theta_v\}} e^{\beta \sum_i \theta_{s(i)}\theta_{t(i)}}}{\sum_{\{\theta_v\}} e^{\beta \sum_i \theta_{s(i)}\theta_{t(i)}}} \\ &= \langle \hat{O} \rangle_{\text{Ising}},\end{aligned}$$

where $O_{\{\theta_v\}}$ is the eigenvalue of $\hat{O}$ when acting on a particular Ising configuration $\{\theta_v\}$. However, we assumed that $\hat{O}$ was made from a combination of σ_i^z operators only. This is important for two reasons: firstly it ensures that $\hat{O}$ commutes with $S(\beta)$ so that we can bring the two factors of $S(\beta)$ together and secondly it means that the Ising configurations are eigenstates of the operator. Now consider the ’t Hooft loop, $\prod_{i\in c} \sigma_i^x$ for a closed contractible dual path c. As the referee points out, it can be written in terms of vertex operators in the region R bounded by c:

$$\prod_{i\in c} \sigma_i^x = \prod_{v\in R} A_v^-,$$

where $A_v^- = \prod_{i\in \text{star}(v)} \sigma_i^x$. In turn, the vertex operators can be written in the Ising basis as spin-flip operations on the Ising spins: $A_v^- = \sigma_v^x$. Notably, this means that the operator keeps the state within the subspace of the toric code model that can be mapped to the Ising model, namely the subspace with no plaquette violations. This means that it is feasible to calculate the expectation value of the ’t Hooft loop in the Ising formalism. This expectation value is given by

$$\langle \hat{C}(c) \rangle_{\text{Deformed Toric Code}} = \frac{\langle GS(0)| S(\beta) \prod_{v\in R} \sigma_v^x S(\beta) |GS(0)\rangle}{\langle GS(0)| S(\beta)^2 |GS(0)\rangle}.$$

Now unlike for the previous case, $\prod_{v\in R} \sigma_v^x$ does not commute with the factor $S(\beta) = \prod_i e^{\beta\sigma_i^z/2} = \prod_i e^{\beta\sigma_{s(i)}^z\sigma_{t(i)}^z/2}$. Consider each factor of $e^{\beta\sigma_{s(i)}^z\sigma_{t(i)}^z/2}$. If R contains both $s(i)$ and $t(i)$ then $\prod_{v\in R} \sigma_v^x$ commutes with this factor (as it introduces two minus signs to the argument of the exponential under commutation, which cancel). Similarly, if R contains neither $s(i)$ nor $t(i)$ then $\prod_{v\in R} \sigma_v^x$ commutes with this factor. This leaves the case where R contains only one of $s(i)$ and $t(i)$, for which the commutation introduces a minus sign to the argument of the exponential factor. This occurs precisely for i in the dual path c. This means that

$$
\begin{aligned}
S(\beta)\hat{C}(c) &= \Big(\prod_i e^{\beta\sigma^z_{s(i)}\sigma^z_{t(i)}/2}\Big)\prod_{v\in R}\sigma^x_v \\
&= \Big(\prod_{v\in R}\sigma^x_v\Big)\Big(\prod_{i\in c} e^{-\beta\sigma^z_{s(i)}\sigma^z_{t(i)}/2}\Big)\Big(\prod_{i\notin c} e^{\beta\sigma^z_{s(i)}\sigma^z_{t(i)}/2}\Big) \\
&= \Big(\prod_{v\in R}\sigma^x_v\Big)\Big(\prod_{i\in c} e^{-\beta\sigma^z_{s(i)}\sigma^z_{t(i)}}\Big)\Big(\prod_{i} e^{\beta\sigma^z_{s(i)}\sigma^z_{t(i)}/2}\Big).
\end{aligned}
$$

This is not surprising, because the same thing occurs if we do not use the Ising basis and instead use the original toric code basis. Then if we write the undeformed ground states in the Ising basis, the expectation value is

$$
\begin{aligned}
\langle\hat{C}(c)\rangle_{\text{Deformed Toric Code}} &= \frac{\sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta'_v\}|\,S(\beta)\prod_{v\in R}\sigma^x_v S(\beta)\,|\{\theta_v\}\rangle}{\sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta'_v\}|\,S(\beta)^2\,|\{\theta_v\}\rangle} \\
&= \frac{\sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta'_v\}|\,\big(\prod_{v\in R}\sigma^x_v\big)\big(\prod_{i\in c}e^{-\beta\sigma^z_{s(i)}\sigma^z_{t(i)}}\big)S(\beta)^2\,|\{\theta_v\}\rangle}{\sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta'_v\}|\,S(\beta)^2\,|\{\theta_v\}\rangle}.
\end{aligned}
$$

Here $S(\beta)^2$ is $\prod_i e^{\beta\sigma^z_{s(i)}\sigma^z_{t(i)}}$, which we can act on the basis state $|\{\theta_v\}\rangle$ to obtain the Boltzmann factor $e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}}$:

$$
\langle\hat{C}(c)\rangle_{\text{Deformed Toric Code}} = \frac{\sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta'_v\}|\,\big(\prod_{v\in R}\sigma^x_v\big)\big(\prod_{i\in c}e^{-\beta\sigma^z_{s(i)}\sigma^z_{t(i)}}\big)e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}}\,|\{\theta_v\}\rangle}{\sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta'_v\}|\,e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}}\,|\{\theta_v\}\rangle}. \tag{1}
$$

In the denominator, we can make use of the orthogonality of the Ising basis states to obtain

$$
\begin{aligned}
\sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta'_v\}|\,e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}}\,|\{\theta_v\}\rangle &= \sum_{\{\theta_v\},\{\theta'_v\}}\delta(\{\theta_v\},\{\theta'_v\})e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}} \\
&= \sum_{\{\theta_v\}} e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}} \\
&= Z(\beta)_{\text{Ising}}.
\end{aligned}
$$

For the numerator, we can also replace $e^{-\beta\sigma^z_{s(i)}\sigma^z_{t(i)}}$ with its eigenvalue $e^{-\beta\theta_{s(i)}\theta_{t(i)}}$. However, there is still an operator $\big(\prod_{v\in R}\sigma^x_v\big)$ which is not diagonal in the Ising basis. However, we know that this operator acting to the right on $\langle\{\theta'_v\}|$ produces a new Ising basis state, with this mapping being one-to-one. Denoting this new basis state by $\langle\{\theta''_v\}|$, the numerator is

$$
\begin{aligned}
\sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta'_v\}|\,\Big(\prod_{v\in R}\sigma^x_v\Big)\Big(\prod_{i\in c}e^{-\beta\sigma^z_{s(i)}\sigma^z_{t(i)}}\Big)e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}}\,|\{\theta_v\}\rangle &= \sum_{\{\theta_v\},\{\theta'_v\}}\langle\{\theta''_v\}|\,\Big(\prod_{i\in c}e^{-\beta\theta_{s(i)}\theta_{t(i)}}\Big)e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}}\,|\{\theta_v\}\rangle \\
&= \sum_{\{\theta_v\},\{\theta''_v\}}\delta(\{\theta''_v\},\{\theta_v\})\Big(\prod_{i\in c}e^{-\beta\theta_{s(i)}\theta_{t(i)}}\Big)e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}},
\end{aligned}
$$

where we changed the sum over $\{\theta'_v\}$ to a sum over $\{\theta''_v\}$ using the one-to-one mapping between them. Then we have

$$
\sum_{\{\theta_v\},\{\theta''_v\}}\delta(\{\theta''_v\},\{\theta_v\})\Big(\prod_{i\in c}e^{-\beta\theta_{s(i)}\theta_{t(i)}}\Big)e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}} = \sum_{\{\theta_v\}}\Big(\prod_{i\in c}e^{-\beta\theta_{s(i)}\theta_{t(i)}}\Big)e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}}.
$$

Substituting the expressions for the numerator and the denominator into Equation 1, the expectation value for the 't Hooft loop is given by

$$
\begin{aligned}
\langle\hat{C}(c)\rangle_{\text{Deformed Toric Code}} &= \frac{\sum_{\{\theta_v\}}\big(\prod_{i\in c}e^{-\beta\theta_{s(i)}\theta_{t(i)}}\big)e^{\sum_i\beta\theta_{s(i)}\theta_{t(i)}}}{Z(\beta)_{\text{Ising}}} \\
&= \langle\Big(\prod_{i\in c}e^{-\beta\theta_{s(i)}\theta_{t(i)}}\Big)\rangle_{\text{Ising}}.
\end{aligned}
$$

We see that even though we mapped the 't Hooft loop on the loop c to the σ_v^x operators in a region R, when we evaluate this expression as an expectation value in the classical Ising model, we return to an expression that depends on the variables near c. While we cannot evaluate this expression, we can obtain limits on it. However, to do this we would perform exactly the same manipulations that we did in Appendix C, except in the Ising basis instead of the loop basis. Because of this, we obtain no advantage from the mapping to classical Ising in this case.