Response to Report 2

January 2024

1 Punctual remarks

1. Introduction. The end of the second paragraph recalls that fracton dynamics in curved geometry is consistent only on certain special backgrounds. The end of the third paragraph seems to aim at a general description of fracton hydro on generic curved background. The two things would clash. Furthermore, in the paper only linear fluctuations about flat backgrounds are studied. The end of the third paragraph seems thus to need revision.

⇒ While it is true that symmetric tensor gauge theories, where the field $A_{\mu\nu}$ is dynamical, are only consistent on certain backgrounds, this is not the case when $A_{\mu\nu}$ is not dynamical. For example, one can construct a scalar field theory with local dipole symmetry that can consistently be coupled to arbitrary Aristotelian backgrounds.

At the same time, we note that the problem of coupling symmetric tensor gauge theories to arbitrary curved backgrounds is related to the fact that the field strength associated with $A_{\mu\nu}$ is not dipole invariant. In the context of this paper we note that the presence of a Goldstone field due to spontaneous breaking of dipole symmetry allows us to define a gauge-invariant symmetric gauge field $\mathcal{A}_{\mu\nu}$ which has an associated dipole invariant field strength.

Regarding the point that we restricted the study of linearised fluctuations to flat backgrounds, we have modified the sentence "The aim of this paper is precisely to formulate different classes of fracton hydrodynamics in generic curved spacetimes and to identify the low energy spectra." to "The aim of this paper is precisely to formulate different classes of fracton hydrodynamics in generic curved spacetimes, and to identify the low energy spectra on flat backgrounds."

2. In the introduction and Subsection 2.4 it is stated that, imposing vanishing U(1) curvature corresponds physically to the absence of elementary dipoles. The statement is argued on the basis of a counting of degrees of freedom and in line with a similar observation made in [JJ22]. I think the statement is not correct. The identification of elementary dipoles is ambiguous and related to improvement transformations of the currents,

similarly to the spin current in the energy momentum for standard field theory. The argument goes like this. The total dipole is

$$\mathrm{d}^i = \int d^3x \, (x^i J^t - J^{ti}) \, . \tag{1}$$

The Ward identities are

$$\begin{aligned} \partial_\mu J^\mu &= 0 \, , \\ \partial_\mu J^{\mu a} &= J^a \, . \end{aligned} \tag{2}$$

Consider the improvement

$$\begin{aligned} \tilde{J}^\mu &= J^\mu + \partial_\nu \chi^{\nu\mu} \, , \\ \tilde{J}^{\mu a} &= J^{\mu a} + \chi^{\mu a} \, , \end{aligned} \tag{3}$$

with $\chi^{\mu\nu} = -\chi^{\nu\mu}$. The improvement respects the Ward identities and allows one to set

$$\begin{aligned} \tilde{J}^{ta} &= 0 \, , \\ \tilde{J}^{[ia]} &= 0 \, . \end{aligned} \tag{4}$$

In particular, the density of elementary dipoles is vanishing.

This seems to solve another issue. In Subsection 2.4, in order to work with $A_{\mu\nu}$ instead of the full $\tilde{A}^a_\mu$, a constraint on the U(1) curvature is imposed. However, according to (2.32), such a constraint would not be gauge invariant. Thereby, relaxing or imposing it should not change the physical content of the theory. Instead, as stated in the last paragraph of the subsection, depending on the enforcing of the constraint, the theory would/would not contain elementary dipoles.

⇒ We thank the referee for these astute observations. The curvature constraint *is* gauge invariant since we assume that the torsion vanishes. We have added a footnote to that effect. The point made about improvement terms is very interesting and also works in curved space in the absence of torsion. To be precise, the relevant part of a generic variation involving $\tilde{A}^\nu_\mu = \tilde{A}^a_\mu e^\nu_a$ instead of $A_{\mu\nu}$ is

$$\delta S \supset \int d^{d+1}x \, e \left[-J^\mu \delta B_\mu + D^\mu{}_\nu \delta \tilde{A}^\nu_\mu \right] \, . \tag{5}$$

The U(1) Ward identity is

$$\nabla_\mu J^\mu = 0 \, , \tag{6}$$

while the dipole shift Ward identity now reads

$$J^\nu h^\mu_\nu - \nabla_\nu D^{\nu\mu} = 0 \, , \tag{7}$$

where

$$D^{\nu\mu} = D^{\nu}{}_{\rho} h^{\rho\mu} \,. \tag{8}$$

Under the suggested improvements

$$\begin{aligned} \tilde{J}^{\mu} &= J^{\mu} + \nabla_{\nu} \chi^{\nu\mu} \,, \\ \tilde{D}^{\mu\nu} &= D^{\mu\nu} + \chi^{\mu\rho} h^{\nu}_{\rho} \,, \end{aligned} \tag{9}$$

the dipole Ward identity remains invariant, i.e.,

$$\tilde{J}^{\nu} h^{\mu}_{\nu} - \nabla_{\nu} \tilde{D}^{\nu\mu} = 0 \,, \tag{10}$$

while the U(1) Ward identity turns into

$$\nabla_{\mu} \tilde{J}^{\mu} = \frac{1}{2} [\nabla_{\mu}, \nabla_{\nu}] \chi^{\nu\mu} = 0 \,, \tag{11}$$

where the last equality relies on the absence of torsion so that, in particular, the relation

$$\nabla_{\mu} X^{\mu} = e^{-1} \partial_{\mu} (e X^{\mu}) \tag{12}$$

holds. In flat space, as the referee points out, the total dipole charge is now

$$\mathrm{d}^{i} = \int d^{d}x \, \left(x^{i} J^{t} - D^{ti} \right) \,, \tag{13}$$

where D^{ti} is the flat space version of $\tau_{\mu} D^{\mu}{}_{\nu}$ and captures the the"internal dipole density". We may, however, choose $\chi^{\mu\nu}$ above such that

$$\tilde{D}^{ti} = 0 \,, \qquad \tilde{D}^{[ij]} = 0 \,, \tag{14}$$

which *removes* the internal dipole density. Imposing the curvature constraint corresponds to choosing the improvement above.

We have included this in the paper in Section 2.4 along with the statement that imposing the curvature constraint corresponds to choosing an improvement term that removes the density of elementary dipoles. In addition, we have included a footnote with special thanks to you, the anonymous referee.

3. Introduction, second paragraph of pag.4. It refers to the consistent definition of the chemical potential. This seems related to the statement below (4.12) where it is said that the *typical ordering* of background gauge field is $\mathcal{O}(1)$. Are these remarks just saying that, if we consider non-trivial chemical potentials, we normally want them to enter at ideal order? The word consistent in the intro seems to refer to something more that I am possibly not getting. Some rephrasing would help.

⇒ All we mean here is that a gradient ordering must be specified for the fields entering the hydrodynamic theory. To avoid confusion, we have removed the word "consistent".

4. In Section 3.1 it would be helpful to say that one turns to Landau grand-potential, otherwise (3.2) seems to clash with comments before it.

⇒ We have added this comment above (3.2).

5. The are some issues in the argument below (3.5). First, it would be helpful to explain the relation $m = 2\partial P/\partial u^2$, maybe connecting to (B.9) and (B.10) (see next point). Then, the argument seems to need to be expressed in a different order, first stating that u^i is invariant and, from that, arguing that m is invariant too. It also seems to lack the extra information coming from the invariance of J^μ under dipole (2.39). Specifically, being ρ and $J^i = \rho u^i$ both invariant, then u^i is invariant.

⇒ We have added some more explanatory text around Eq. (3.5). In particular, we now explicitly write the Gibbs–Duhem relation, which explains the relation $m = 2\partial P/\partial u^2$. We thank the referee for pointing out that the order of the argument should be reversed; this has now been done. The argument for the dipole invariance of u^i relies on the fact that it is part of a thermal vector defined in terms of a background Killing vector (in the general case) which does not transform under dipole transformations. However, another way to arrive at the same conclusion is to argue that J^μ does not transform, as the referee suggests. We have included both arguments in the text.

6. There seems to be a source of confusion in some adopted notations. In Section 3.1, u represents the frame velocity, namely the spatial equivalent of a chemical potential. In Appendix B, the 4-vector containing the chemical potential and the velocity components is instead indicated with ξ, while u assumes a dual meaning according to (B.6) and (B.7).

⇒ In Appendix B, we treat both the relativistic and the Aristotelian superfluids, which have a broken U(1) symmetry. This implies the presence of a Goldstone field ϕ, which allows us to construct the "superfluid velocity" $\xi_\mu = A_\mu - \partial_\mu\phi$. This is very different from the actual velocity u^μ, which is part of the thermal vector k^μ, which is a Killing vector of the background. For Aristotelian fluids (both super and normal), where there is no boost-invariance, physical quantities will explicitly depend on the chosen frame, which is reflected in the presence of an additional chemical potential $\vec{u}^2$ for the kinetic mass density m. In other words, ξ_μ and u^μ are very different, and u^μ only gives rise to an additional chemical potential in the Aristotelian case.

7. Introduction. The description of quadratic dispersion relations is confusing. First, the coefficient a entering a dispersion relation as $\omega = ak^2 + \dots$ is systematically referred to as "velocity" but it is not a speed at the dimensional level. Secondly, the term magnon suggests physical aspects which are not present here. Magnons are spin-waves whose quadratic behavior is related to a lack of time-reversal symmetry. Here neither spin nor time

reversal is concerned. As far as I am understanding "magnon-like" is used just as a synonym of "quadratic" which would be preferable.

⇒ It is true that "magnon-like" just means quadratic and that "magnon velocity" refers to the coefficient appearing with k^2. This nomenclature has (perhaps unfortunately) become standard in the literature so we have used it here also. In any case, we have added footnote 4 to specify what we meant by this terminology.

8. Introduction, fig.1. Why does not the pinning translate into a gap for a mode? This is the customary sense of the word "pinning" in condensed matter. Pinned charge-density-waves, for example, have a gapped sliding mode.

⇒ We used the word "pinning" in analogy with condensed matter systems, but the origin of the pinning we consider is different than in typical setups. In particular, in the context of charge density waves, crystals, etc, pinning arises due to explicit symmetry breaking while in the context of s-wave fracton superfluids no symmetry is explicitly broken. Nevertheless one can still write a kind of "mass term" that involves a combination of Goldstone fields. In any case we should not expect the same phenomenology. We have added footnote 10 with this explanation.

9. The derivation of (2.38) is not explained clearly. It seems that (A.3) is used without saying and possibly also the Ward identities (2.41) derived only later.

⇒ We have rewritten the relevant paragraph slightly so it is clear that those equations were used.

10. The derivation of (B.5) is not easy to follow. First, the same symbol ξ is used for the superfluid velocity and for the vector generating a generic diffeomorphism. (B.5) is a consequence of the last of (B.4) and its variation, $\delta\delta_K\phi = 0$ (this could be said to help the reader). However, the meaning of δ in (B.5) is different from that just used in (B.1), in fact one is using δ in the sense of $\delta\phi = \pounds_\xi\phi + \sigma$, comprehending both a diffeo and a gauge transformation.

    This said, I find it not clear how (3.9) could descend from (3.8) as suggested by the text between them. The last of (3.9) is analogous to (B.5) which needed the last of (B.4), absent in (3.8).

    The second of (3.9) is got from $\delta\delta_K B_\mu = 0$ using the second of (3.8) and the last of (3.9). It could be useful to explain more explicitly how to get (3.9).

⇒ We have added some additional explanatory text around (B.5), and we have changed the name of the infinitesimal diffeomorphism from ξ to ζ to avoid confusion with the superfluid velocity ξ_μ.

We have also added additional text around (3.9) (which is now (3.10)) to explain how it is derived.

11. Subsection B.1.1. In (B.6) the parameter T_0 is introduced, but then is considered only $T_0 = 1$ without saying. Eq. (B.7) is actually the definition giving rise to the third of (B.6), the phrasing between (B.6) and (B.7) is confusing.

⇒ The constant "global" temperature T_0 that features in the definition of T is not set to one in that section, but it does not feature explicitly in the other expressions. However, it implicitly appears in all instances of T.

From the perspective of the hydrostatic partition function, the name of the game is to construct all gauge-invariant scalars, and ξ^2 is one of those. Another is $\mu = u^\mu A_\mu + T\sigma^K$, while a third is given by $u^\mu \xi_\mu$. However, this is the *same* as μ in equilibrium.

12. Plugging (B.9) into (B.14) and differenciating, there is a problem with the sign of the term in $d\xi^2$. Does this propagate to (B.13)?

⇒ Well spotted! The sign in the expression for the first law is a typo and does not affect the other equations (cf., pp. 2–3 of this paper by Jensen et. al.)

13. In (5.10) there seems to be a typo, θ has not been introduced before. It is probably a ϕ and, using an expression analogous to that given below (B.7), it justifies the statement $\mu_p = u^\mu \mathcal{B}_\mu$ given just afterwards. Some more in-line explanation would be useful.

⇒ This is indeed a typo. We have added a sentence that says that the identification $\mu_p = u^\mu \mathcal{B}_\mu$ is a result of the equilibrium condition on ϕ.

14. The constraint on charge mobility due to dipole symmetry is valid only when a charge is isolated and, in particular, when it cannot exchange dipoles with a background or a dipole condensate. Why then should the p-wave fracton superfluid feature a sort of no-flow theorem like that emerging from (4.17)? Relatedly, since both the p- and s-wave fracton superfluids are on the same footing as far as the dipole symmetry and its breaking are concerned, why should there be such a qualitative phenomenological difference about the possibility to flow? I believe these points need to be discussed more in the paper.

⇒ We emphasise that (4.17) is not related to the no-flow theorem that we discuss in §3.1. Rather, it is a consequence of the gradient ordering that we impose and implies that the p-wave superfluid has fluid velocity that is $\mathcal{O}(\partial)$ and hence it can flow but slowly. The same choice of gradient

ordering can be made for s-wave superfluids, which we discuss in Appendix C, but as we also argue throughout the paper we think that this choice of gradient ordering for the s-wave superfluids is not the most appropriate one. The reason why gradient orderings can be different in the two phases is simply because the field content in the two phases is different.

15. Why does (5.1) feature a minus sign with respect to an analogous equation in (B.1)?

⇒ This is a choice. Appendix B was written to conform to the conventions usually adopted in the literature on superfluids. We have added a footnote about this in Appendix B.

16. The comparison of the term proportional to the scalar Goldstone variation between (5.25) and (B.18) seems problematic. In (5.25) there appear a calligraphic kappa, while in (B.18) there is a normal K.

⇒ We are unsure which comparison the referee refers to here. We do not compare the two expressions in the paper. They are the adiabaticity equations for two separate hydrodynamical systems. The reason why (B.18) has a different sign is related to how K is defined in (B.11), which has the opposite sign compared to (5.3).

17. The steps at pag.29 seem to me correct, however I have found it hard to follow them. Maybe some effort to streamline would be useful.

⇒ The calculations on page 29 show how our results relate to those of Głódkowski, Benítez and Surówka, and in particular show that the constitutive relations they find are exactly those of the flat-space p-wave fracton superfluid that we develop. We are not sure which parts are particularly confusing: to us, the section accomplishes its aim in that it demonstrates the equivalence of the constitutive relations. Nevertheless we tried to improve it by adding various comments and details. If the referee is unsatisfied with parts of it, we would be grateful if the referee could point out the particular parts of it that appear to be confusing.

18. The identification $p_j = -n\Psi_j$ in (4.45) is commented at pag.30 but it appears to be confusing. Momentum as a physical quantity should be gauge invariant, a Goldstone field is not gauge invariant.

⇒ Indeed this is one of the main differences between fracton theories and other theories, namely the fracton algebra implies that momentum transforms with dipole transformations as pointed out in various papers, including Ref. [23]. This is also clearly visible from the second variation of the action, which states that both the energy current and the spatial stresses in (2.48) must transform under dipole transformations. By defining momentum as usual $p_\mu = \tau_\mu T^\mu{}_\lambda h^\lambda_\mu \sim T^0_i$ we clearly see that it must transform under dipole transformations. This explains, for instance, the

identification provided in (4.45). We note, however, that for p-wave superfluids there is a definition of spatial stresses that allows for a notion of dipole-invariant momemtum, namely that given by (4.8).

19. (5.21) is analogous to (B.13). What is the analog of (B.12), which one should use to get to (5.21)?

⇒ The hydrostatic superfluid equations of motion in (5.21) are obtained by varying the "ideal" order pressure $\mathcal{P}$ defined in (5.17). The responses are defined in (5.18). To more clearly showcase the origin of the expressions in (5.21), we have included the variation through which they are defined. We hope this makes their derivation sufficiently clear to the referee.

20. Well-defined gradient expansion and UV/IR mixing. The parts within Sections 5.2, 5.4 and Section 6 that are concerned with the gradient expansion are confusing. Apparently, having $A_{\mu\nu} \sim \mathcal{O}(\partial)$ gives the hydrodynamic series a structure of double expansion in both the wave-number k and $A_{\mu\nu}$. From the comments given in the manuscript, it is not clear whether one is free of considering different regimes like $A_{\mu\nu} \ll k$ or $A_{\mu\nu} \gg k$, or if one is obliged to consider $A_{\mu\nu} \sim k$. Above (5.17), it is stated that higher order scalars built from $A_{\mu\nu}$ can affect lower orders in wavenumbers. This would relate to $A_{\mu\nu} \gg k$ and has the downside of making the hydro expansion completely unpredictive. This is in line also with the comments given above (5.33) where, including higher powers of ξ, can modify the lower order coefficients in k in the dispersion relations. In the page change from 47 to 48, however, it is said that the gradient expansion followed in the main text reconciles the gradient expansion with that in wave-number, therefore $A_{\mu\nu} \sim k$. Contrasting with the subsequent sentences where it is said that higher derivative terms also affect lowers orders in k. All in all, it is not clear whether one could just take an expansion in orders of ∂.

    Note that at the end of pag.47 the gradient expansion done for the s-wave with the same scaling hypothesis as the p-wave is discarted precisely because of possible "mixing effects" among the orders, when indicating that the higher-derivative term related to m affects the IR-regime.

⇒ This was an interesting comment and it took us some time to understand a bit further the structure of linear modes. We begin by noting that the gauge invariant $\mathcal{A}_{\mu\nu}$ indeed includes the background gauge field $A_{\mu\nu}$ and hence the gradient scheme $\mathcal{A}_{\mu\nu} \sim A_{\mu\nu} \sim \mathcal{O}(1)$ requires us to only couple the fluid to $\mathcal{A}_{\mu\nu}$ perturbatively. In other words, imposing this gradient scheme implies that s-wave flows must have small dipole superfluid velocities which in the context of the couplings we considered is equivalent to expanding in small dipole superfluid density. Admittedly, we do not think that this was extremely clear in the earlier draft so we added some comments below equation (5.8) while above (5.17) we introduced a book-keeping parameter ε to control the strength of the couplings to $\mathcal{A}_{\mu\nu}$.

Going back to the referee's question, we are free to choose the relative strength of ε with respect to the momentum k and consider all possible regimes $\varepsilon \ll k$, $\varepsilon \sim k$ and $\varepsilon \gg k$ as long as ε is treated perturbatively. In the earlier draft we had considered $\varepsilon \gg k$, as the referee pointed out, but because the couplings must be treated perturbatively we show that this regime is equivalent to $\varepsilon \sim k$. We have restructured Section 5.4 with the various mode calculations and expansions in ε as well as given a summary of the structure of modes at the end of Section 5.4. In that summary we also compare these linear modes with the modes obtained using the alternative gradient scheme. We argue in that summary that while one scheme appears to show signatures of UV/IR mixing, the other does not at least for the corrections and the equilibrium states that we considered. In line with Section 5.4 we adapted the discussion of the gradient schemes in Section 6 which we hope will not cause further misunderstandings. We feel that this analysis of the gradient and momentum expansion is currently as exhaustive as it can be without increasing significantly the scope of the paper.

2 Curiosities

1. In the paper, only fractonic symmetries associated to multipolar symmetries are considered, what about subsystem symmetries?

⇒ We find this an interesting direction and it should be possible to gauge subsystem symmetries and obtain a curved spacetime geometry as in Section 2 of our paper. If this can be done consistently, one could aim at developing hydrodynamic theories for subsystem symmetries. We have added a comment on this at the end of Section 6 along with a few references.

2. The mass/pinning of the Goldstone in the s-wave fracton superfluid is very interesting. What is the relation of such pinning to the Goldstone counting problem and/or inverse Higgs constraints?

⇒ We haven't investigated this in detail and we do not have, at this point, anything non-trivial to say except that there must be a relation with inverse Higgs constraints since in the U(1) regime of s-wave superfluids we find the condition $\Psi_\mu = h^\nu_\mu(B_\nu + \partial_\nu\phi)$ which appears to be similar to typical inverse Higgs constraints. We have added footnote 20 as well as a new reference.

3 Typos

1. The indices of the $\tilde{A}e$ combinations in (2.30) are wrong.

⇒ We thank the referee for pointing out this typo; this has now been fixed.

2. Last paragraph of 2.4, the removed components are $d(d-1)/2$ and not $d(d+1)/2$.

$\Rightarrow$ We disagree with this comment. The $U(1)$ curvature is an antisymmetric tensor in $(d+1)$ spacetime dimensions (and not d dimensions), and so enforces $d(d+1)/2$ conditions.

3. First sentence of Subsection 2.6, "which" $\rightarrow$ "in which".

$\Rightarrow$ We thank the referee for pointing out this typo; this has now been fixed.

4. Between (5.14) and (5.15), "thos" $\rightarrow$ "those".

$\Rightarrow$ We thank the referee for pointing out this typo, this has now been fixed.

5. Just before (4.27), "must a" $\rightarrow$ "must be a".

$\Rightarrow$ We thank the referee for pointing out this typo, this has now been fixed.

6. Before (4.47), the transformation $f_s \rightarrow -f_s$ is referred to as a "shift", but it is not.

$\Rightarrow$ We thank the referee for pointing out this typo. "Shift" has now been replaced with "redefinition".

7. Text between (5.18) and (5.19). "second variation (5.3)" $\rightarrow$ "second variation of (5.3)".

$\Rightarrow$ We thank the referee for pointing out this typo, this has now been fixed.

8. Below (5.21), "as the for p-wave" $\rightarrow$ "as the one for p-wave".

$\Rightarrow$ We thank the referee for pointing out this typo, this has now been fixed.