Replies to main comments:

1. The main statement of this paper involves line operators in 4d Chern- Simons theory, yet, such operators are described only in terms of string theory brane configurations. To discuss line operators in a gauge theory, they should be defined in terms of that very gauge theory. The names for the line operator examples in the last section (such as Wilson lines, etc.) are suggestive, however, the operators are still left undefined.

It is accurate to say that we have not defined the line operators in 4d Chern-Simons theory in the way line operators in gauge theories are usually defined. For example, it is common to define Wilson lines and 't Hooft lines as the holonomy along a line and as certain singular profile of the gauge field respectively. Instead of trying to give such definitions for the family of line operators we consider, we have provided a string theoretic definition. This is mainly because even our definition of the 4d Chern-Simons theory is primarily in terms of branes, as the world volume theory of D5 branes in an Ω-background. So we believe it is justified to also define the line operators using their brane constructions. That being said, we do briefly give a somewhat indirect definition of the line operators purely in gauge theory – as certain quantum mechanics coupled to the 4d Chern-Simons (Introduction, 2nd paragraph). Definitions of Wilson and 't Hooft lines as certain quantum mechanics is not new [1–4], for the more general line operators considered in our paper, the more standard gauge theoretic formulation is not clear to us and did not play any role in the paper.

2. Pages 5-6 and 15: There is an issue with two different interpretations of the Ω-twisted brane configuration. Note, that Ω-twist is simply a statement of that the string ambient space is. Namely, Eq. (4.1) implies that (4,5,6,7) directions are cotangent to the (0,1,2,3) directions. This interpretation follows from the choice of the Ω-twist and is essential for the whole reasoning of this paper. Alas, it is in contrast with the geometry presented in the beginning of the paper, where (3,4) directions are cotangent to (0,7), while (1,2) and (5,6) form the Taub-NUT space. This latter was used to have a 4d Chern-Simons formulation. The authors should reconcile these two formulations. If this can only be dome when $\Sigma = \mathbb{C}$ and $C = \mathbb{R}^2$, then there is no use of introducing Σ and C, which suggest that these results could be applied to more general setup.

To study the phase space or the Hilbert space associated to a spatial slice N in a space-time M it suffices to study the equations of motion of the theory on a collar near N, i.e., we can consider the theory on $N \times \mathbb{R}$. This is essentially what is being used implicitly. The line operators in the 4d CS theory are parallel to the topological plane Σ and local in C. So, for the purpose of computing phase spaces, we can focus on the 4d space-time $S^1 \times \mathbb{R} \times C$ where $\mathbb{R}$ is the direction in which the line operators are extended. In this setting the roles of $\mathbb{R}$ and its cotangent direction are interchangeable. The local (in space-time) analysis of the supersymmetric twists from the 6d (D5 brane world-volume) and the 4d (D3 brane world-volume) was done in more detail in [3, §6.2].

3. P. 21, above Eq. (4.46), "we find that once Ω-deformation is turned on": This is a sudden big step. In contrast to all the details in the preceding sections, Eq. (4.46) appears with little justification.

This statement relies on the following older result: If Q is an Ω-deformation supercharge in a 2d B-model with superpotential W, then localizing with respect to Q gives a 0d theory with action W (up to proportionality factors). In the specific context of our paper (mainly in §4.1.1 and §4.1.2) we write the 4d Kapustin-Witten action as a 2d B-model action (the two "transverse" directions being thought of as "internal") whose superpotential is identified with the 2d BF action in the transverse directions. It then follows that when localized with respect to an Ω-deformation supercharge, the 4d KW theory localizes to a 2d BF theory.

Reference for deriving two lower dimensional actions from Ω-deformed theories added.

Replies to minor issues:

- **Page 2, paragraph 2:** Coupling a gauge theory with gauge group G to a system with symmetry H requires specifying the action of G on that system, thus if G is a subgroup of H, ($G \subseteq H$) it is straightforward. The inclusion the authors state, however, is in the other direction with $G \supseteq H$. It is far from clear how a system with lower symmetry can be coupled to a gauge theory, without first reducing the gauge group (which, in quantum field theory, requires a lot of extra care). This is probably a superficial issue, since the coupling is described at the end of Sec. 2.1. However, some clear language in the introduction is in order.

  What we need is a homomorphism $G \to H$ by which G acts on the phase space of the quantum mechanics. Statement in the introduction modified to reflect this.

- **P.4, line 3:** "in principle" instead of "in principal".

  Corrected.

- **P.5, after Eq. (2.1):** $\vec{\omega}$ is a vector field on $\mathbb{R}^3\setminus$ (a line). This vector field is NOT globally defined of all on $\mathbb{R}^3$, nor is the circle fiber coordinate θ.

  Support of $\vec{\omega}$ corrected. The TN circle coordinate not being well-defined on all of $\mathbb{R}^3$ is implied by the TN circle collapsing at the center in the next line.

- **P.5, par. -3:** the 1d TQM statement is not justified. Please provide a reference or an explanation.

  References added.

- **P.6, par. 1:** state that K_i and L_i are integer-valued.

  Stated.

- **P. 15, title of Sec. 4.1:** "as a 2d B-model" instead of "as A 2d B-model".

  Corrected.

- **P. 18, Sec. 4.1.2,** "the following inner product": where is this inner product valued? Why is (4.25) finite?

  It is valued in $\mathbb{C}$ when it is finite. It is not necessarily finite for all fields, just as an action for a QFT is not necessarily finite for all fields. Finiteness of actions (the inner product becomes part of the action, as in eq. 4.26) is a tricky issue that we do not address in the paper.

- **P. 22, line 2 after Fig. 4:** "nth D5 brane" instead of "ith D5 brane." Also, one line above K_n instead of K_i.

  Same par: "under" instead of "udner."

  n was used as the total number of D5 branes whereas "ith D5 brane" refers to any one of the D5 branes, $i \in \{1, \cdots, n\}$. Similarly K_n is the linking number of the very last D5 brane, whereas K_i is linking number of the ith D5 brane where i can be any integer between 1 and n (inclusive).

  Spelling corrected.

- **P. 35: top par:** Is this statement that turning on the twisted masses corresponds to this particular quantization of the universal enveloping algebra a conjecture? If not, what justifies it? Can one derive the corresponding deformed structure relations?

  This is not a conjecture. An isomorphism can be found in Example 6.2 in [5]. Reference is added in the latest version.

- **P. 37, Remark 5.3:** "If we unsymmetrize ..." this is unclear. What is unsymmetrization? What is this unsymmetrization applied to? It seems that it could not be the Higgs branchs, so is this an operation on a direct sum of the corresponding algebras? This remark is not clear.

  Unsymmetrization is applied to the Coulomb branch algebra. Unsymmetrization of an algebra A (assuming containing $\mathbb{C}[r_1, \cdots, r_n, \hbar]^{S_n}$) means the operation that adds non-symmetric polynomials of variables $r_1, \cdots, r_n$ to A, i.e.

  $$A \rightsquigarrow A \otimes_{\mathbb{C}[r_1, \cdots, r_n, \hbar]^{S_n}} \mathbb{C}[r_1, \cdots, r_n, \hbar].$$

  We change the phrasing in the latest version accordinglly.

References

[1] J. Gomis and F. Passerini, *Holographic Wilson loops*, *JHEP* (2006) 074, 30 [`hep-th/0604007`].

[2] J. Gomis and F. Passerini, *Wilson loops as D3-branes*, *JHEP* (2007) 097, 12 [hep-th/0612022].

[3] N. Ishtiaque, S. Faroogh Moosavian and Y. Zhou, *Topological holography: The example of the D2-D4 brane system*, *SciPost Phys.* **9** (2020) 017 [1809.00372].

[4] K. Costello, D. Gaiotto and J. Yagi, *Q-operators are 't Hooft lines*, 2103.01835.

[5] S.F. Moosavian and Y. Zhou, *Towards the Finite-N Twisted Holography from the Geometry of Phase Space*, 2111.06876.