Report on "Virasoro Minimal String"

January 22, 2024

We would like to thank the referee for their thoughtful comments and constructive remarks. They are much appreciated. We would like to offer some comments to the points of the referee.

1 Anonymous Report

1. I think calling this model the "Virasoro minimal string" is unnecessarily misleading, since it has nothing to do with the minimal string. The authors mention this point in footnote 5, but I still think, given the names of earlier models, this is an unfortunate and confusing name.

   As far as we understand the complaint is mostly about our use of 'minimal' and we can see both the argument for including or not including it in the title. We discussed this at length and think that this is a matter of taste. The theory is minimal as mentioned in footnote 2 of the paper. We decided to include 'minimal' in the end for this reason and would rather not change it at this point since it is already partially ingrained in the community.

2. The earlier work 1704.07410 is also relevant for the translation of the sinh dilaton model to the Liouville description.

   We thank the refereee for pointing out this reference and we added it alongside the other references stressing this relation.

3. The authors mention in footnote 3 that earlier works focus solely on the minimal string. This is not correct. Some references (e.g. ref 38) work with general Liouville gravity, where any matter CFT is coupled to the usual spacelike Liouville and ghost CFTs. For the higher genus multiboundary analysis, the minimal string was explicitly considered in those works in order to have concrete matrix model descriptions, but most of the other results deal

with generic Liouville gravity models. E.g. the correlators with boundary tachyon vertex operators were argued to hold for any choice of matter sector, where in fact the minimal string boundary correlators were discussed to have a special "degenerate" structure (as discussed separately in 2007.00998 by one of those authors). The claims there would be that these are also applicable to the current situation of timelike Liouville CFT as the matter sector.

We thank the referee for their comment. We agree that some of the results of 2007.00998 hold for arbitrary matter theories. However, these results are for quantities that we don't consider in this paper, such as boundary two- and three-point functions. These quantities do not involve any integration over moduli and thus the matter contribution factors out and it is quite simple to consider an arbitrary matter theory. Clearly, the statement that there is a matrix model description for any matter theory is incorrect. For instance, a matter system of two free bosons coupled to spacelike Liouville CFT would serve as a counterexample. Thus the results in 2007.00998 are of limited applicability in this context. We highlight in particular that to obtain the quantum volumes beyond the one for the three-punctured sphere we need to perform integrals over moduli space which depend sensitively on the details of timelike Liouville theory. Thus we stand in essence by the content of the footnote, but removed the reference to Liouville gravity in particular.

4. The main question and concern I have is about the precise density of states of the model in eq. (1.3). The authors do not explicitly discuss this, but with their choice of asymptotic boundary condition in eq (8.2), the disc partition function is essentially the Virasoro vacuum character (without the secondaries). This implies a Cardy scaling and in particular, a JT-like density of states in the thermodynamic limit: $T \sim \sqrt{E}$. However, the classical black hole solution in this dilaton gravity model has a thermodynamic relation of the form $T \sim \sqrt{E^2 - 1}$ (in suitable units) as discussed in ref. 38 using general results on dilaton gravity models. The classical black hole solution for the hyperbolic sine dilaton potential has the property that far away it deviates from AdS whereas deep in the interior it is AdS (and hence JT-like). This suggests that a "conformal" $T(E)$ relation (as in JT and as in the vacuum Virasoro character), is not a priori expected. Perhaps one can get away with this by e.g. changing the time coordinate w.r.t. which one defines thermodynamics and/or choosing different boundary terms/conditions, but this is not a priori clear. It would be useful if the authors can clarify this point.

Relatedly, the transformation (8.2) the authors use deviates from that used in the past by ZZ and used by all papers (I am aware of) that build on this. The difference is essentially the energy variable being s^2 instead of $\mu_B \sim \cosh(2\pi bs)$, the boundary cosmological constant of the FZZT brane. This is precisely the reason for their density of states (1.3) taking the form it does, and it not being the same as that found for generic Liouville gravity models in the literature. I would ask the authors to elaborate on where equation (8.2) comes from precisely, purely from the worldsheet or dilaton gravity perspective.

We agree that it goes against the earlier literature that we identify the boundary energy as $E \sim s^2$ instead of $\cosh(2\pi bs)$ and already remarked on this in section 8.1 before. However, we agree that this point deserves some more explanation.

From an abstract point of view we should note that the worldsheet definition of the theory does not have this information built in. Indeed, there should be a boundary condition on the worldsheet for any operator $\text{tr} f(H)$ in the matrix model, where f is a reasonably nice function of the Hamiltonian H. Thus one can of course define a boundary condition where one replaces s^2 by $\cosh(2\pi bs)$, but our claim is that it does not map to the partition function in the matrix model. This transformation is part of the dictionary we propose. It is unambiguously fixed by computing the spectral curve from the topological recursion that we derive (which does not require the introduction of boundaries) and comparing with the disk partition function. Thus we have no doubt that this is the correct boundary condition (and we check that it correctly reproduces the double trumpet in addition to the trumpet partition function, which is a non-trivial check of the proposal).

The usual motivation of the identification of the energy E with $\cosh(2\pi bs)$ does not directly carry over to this case. The argument is that one wants to introduce a fixed-length boundary condition, which one obtains by Laplace transforming with respect to $\mu_\text{B} \sim \cosh(2\pi bs)$. In this case, the metric both involves the spacelike and the timelike Liouville field. We do not understand the boundary condition that we propose for the timelike Liouville factor from a path integral perspective, and thus we currently lack the technology to provide a derivation directly from the path integral.

We find it however quite natural that s^2 is identified with the energy. Indeed, by definition the energy in the Virasoro algebra is defined in terms of the eigenvalue of L_0. The FZZT parameter is defined such that s is the Liouville momentum in the open-string channel of the annulus, and hence the energy is s^2.

We don't have a good answer for the comment about the thermodynamic relation from the dilaton gravity perspective, and thank the referee for raising it. In this work we only used the dilaton gravity model as a motivation for the rest of the paper. The essential aspects of the duality — which, in our view, has been quite convincingly demonstrated in this paper — do not involve the path integral formulation of the theory in terms of sinh dilaton gravity. It would be very interesting to better understand the relation between the Virasoro minimal string and sinh dilaton gravity beyond the level of the classical action, however it is beyond the scope of this work.

We added some more comments on the relation of energy and the FZZT parameter in section 8.1 along the lines mentioned above.

In this light, I also think section 8.1 is quite important but its placement in the paper does not suggest this. Perhaps it could be useful to have this discussion earlier on.

This is again a matter of taste and we had long discussions about how to best organize the paper. Section 8 depends on almost all the previous sections and thus it is quite hard to place it earlier without heavily modifying the structure of the paper. It is also the most technical part of the paper and would be difficult to fully explain its contents earlier.

5. In the discussion section 9, the authors discuss the potential application to near-extremal black holes. Given the previous comments, this again is confusing to me. The sinh dilaton gravity black hole solution does not look at all like the BTZ black hole, except in the JT limit. The Ricci scalar of the sinh dilaton black hole rises without bound as one goes further away from the horizon. This is a property not found for the BTZ black hole (which has constant value of the Ricci scalar). Focussing instead then on the near-horizon region we get back to just JT gravity. Do the authors have any additional insight in this proposed regime of the BTZ black hole?

   We have demonstrated that the relationship between near-extremal BTZ and VMS holds at the level of the partition function. Upon taking, say, the left-moving temperature to be small, we freeze out the left-movers and are left with a single Virasoro vacuum character, which is indeed the disk partition function of the model. For our purposes, the description of the model in terms of sinh dilaton gravity was a motivation for this work and we have not studied it on its own terms in detail. Thus we have nothing more to say on this from the dilaton gravity perspective (although we thank the referee for highlighting the tension between the black hole in sinh dilaton gravity and the BTZ black hole.)

6. Whereas the authors give ample references in their work, sometimes those are not necessarily placed everywhere where they are important. E.g. equation (8.1) is well-known from mainly Teschner's work on quantum Teichmüller space and Liouville CFT, so a reference to these earlier works at this point can be useful.

   We have added a reference of Teschner's work to equation (8.1).