Report on "Virasoro Minimal String"

January 22, 2024

We would like to thank the referee for their thoughtful comments and constructive remarks. They are much appreciated. We would like to offer some comments to the points of the referee.

Report by Sylvain Ribault

Comments about substance

1. *The idea of coupling two spacelike Liouville theories comes a bit late on page 89, when it might be relevant to the speculation about analytic continuation in c in Section 4.5, and to leaving the $c \geq 25$ regime as discussed page 82. It would be nice to tie these points together.*

   The idea of coupling two spacelike Liouville theories is indeed a very intriguing set-up and we are currently exploring it. However this theory is completely disconnected from the VMS (in particular the string diagrams of the two theories are not simply related by analytic continuation) and hence we only mention it in the discussion.

2. *The sense in which the two- and three-point structure constants are universal, as discussed on page 19, could be made more precise. From the point of view of the conformal bootstrap, these quantities suffer from the ambiguity of renormalizing the primary fields $V_P \to \lambda(P)V_P$ in each Liouville factor. (Not to be confused with the normalization in (2.6).) Does the quantity $\rho_0^{(b)}$ have an intrinsic, normalization-independent definition in CFT? Or does the 0 relation with the matrix integral somehow distinguish a particular normalization in Liouville theory?*

   As described below equation (3.3) this normalisation is distinguished in the sense that both the two- and three-point functions involve crossing kernels for vacuum conformal blocks, and hence are related to universal properties of compact 2d CFTs. See in particular equation

(2.10) for the intrinsic definition of $\rho_0^{(b)}$ as a modular crossing kernel in 2d CFT. The relation with the matrix integral only picks out a normalization of the combined vertex operators (2.6), such that the sphere three-point diagram is normalized to 1.

3. *As stated on page 25, it is certainly interesting to compare (3.16) with the leading behaviour of ω_{WP} . Having done the comparison, what do we learn? There is probably the technical statement that the limits $z \to 0$ and $b \to 0$ do not commute, is there something more conceptual to say?*

   This was intended as a loose end comment. We believe that it is interesting to compare with the leading behaviour of ω_{WP} although we do not have much more to say about it than what is already written on page 25. It is not clear to us whether the limits in z and b commute or not. If the referee prefers we could also remove this paragraph since it is interesting but slightly inconclusive.

4. *The interpretation and notation for the half-ZZ branes are not too clear. According to the wavefunctions, these look like $(m, 0)$ or $(0, n)$ ZZ branes, or alternatively like derivatives of an FZZT brane at certain values of their parameter. Quite confusingly they are called $(m, \pm)$, and even compared to $(m\infty)$ and (∞, n) ZZ branes. It might be useful to study 1 these branes in spacelike Liouville theory. (The divergence observed in footnote 14 might make them more FZZT-like than ZZ-like?)*

   The $(m, 0)$ and $(0, n)$ ZZ boundary conditions do not make sense in spacelike Liouville theory. It was unclear to us how to relate the half-ZZ boundary conditions to the more conventional ZZ boundary conditions. As remarked in footnote 14 the half-ZZ boundary conditions do not seem to make sense in spacelike Liouville theory due to a divergence in the cylinder partition function (here there is no contour to adjust to try to make sense of it). Moreover the ordinary ZZ boundary conditions seem less natural in timelike Liouville theory, for which the degenerate representations are on slightly different footing than in the spacelike theory.

5. *In Section 7, I am a bit skeptical that the advantage of having Barnes G-functions justifies the pain of having b^2 almost rational in computing blocks, and the ensuing loss of generality in the choice of the central charge. As is clear in (C.8), we are replacing one Γ_b with a number of Gs that depends on b^2. But Γ_b is not that hard to compute numerically: either by the integral formula (in which case it is advantageous to compute a product of several*

Γ_bs as one integral) or by the product formula of Alexanian and Kuznetsov. Moreover, I am surprised by the discussion (footnote 19) about which integral to perform first in the triple integral: aren't multiple integrals faster to compute together rather than sequentially?

This is an interesting comment and the referee might be right that it could be more convenient to use the product formula. For the purposes of the VMS computations that appeared in the paper, restricting to rational b^2 and using the representation in terms of Barnes-G functions was simply a matter of convenience. Even with this added pain in evaluating the conformal blocks the evaluation of the string diagrams only took on the order of a few hours.

6. *Is the code for Section 7 publicly available? If not the reader would have to take the authors' word for that section's results. (The displayed curves would be very easy to generate without doing the hard work.) I do not consider this a major issue since there is already ample evidence for the main claims in the rest of the article.*

   The code for this paper is not publicly available. However for the limiting case $c = 25$ the code for the torus one-point and sphere four-point function is avalaible in 2302.06625 and 2304.13043. Furthermore all the detailed formulas are provided in section 7 and appendix C, which combined with the available notebooks for $c = 25$ should allow the reader to reproduce our results. We are happy to make the notebook available upon request.

7. *The prescription (8.3) looks ad hoc. Basically, we divide by a factor such that eventually in (8.5) and (8.13) nothing depends on the boundary state in the timelike Liouville theory. Is this cancellation specific to half-ZZ branes? Or could we have chosen any boundary state in timelike Liouville theory? This phenomenon is ascribed to BRST equivalence, but it is strange that BRST equivalence would apply the timelike Liouville factor alone. This is not what the equivalence is supposed to do in the case of ZZ-instantons (8.19). These two behaviours ascribed to BRST equivalence do not seem consistent with one another, given that in ordinary spacelike Liouville theory, a ZZ-brane is a linear combination of two FZZT-branes.*

   It is true that the prescription outlined in (8.3) is in some sense ad hoc. But it is nevertheless non-trivial that this prescription works and satisfies additional consistency checks. Let us first mention that historically, we actually discovered the half-ZZ boundary conditions when attempting to reproduce non-perturbative effects in the matrix integral from the worldsheet.

We then formulated the boundary conditions corresponding to asymptotic boundaries by combining FZZT branes for the spacelike theory with these half-ZZ branes for the timelike theory. While it is true that the prescription (8.3) is essentially designed to reproduce the trumpet and disk partition functions of the matrix model, it is non-trivial that the (marked) cylinder diagram correctly reproduces the double-trumpet (see the computation on page 76). For example this does not work if you replace the half-ZZ boundary condition with the ordinary ZZ boundary condition for the timelike sector (together with an appropriate marking prescription). In general, conformal boundary conditions for timelike Liouville CFT have not been explored much in the literature, and a more thorough exploration is beyond the scope of this work.

Let us also mention that a similarly ad hoc marking prescription is commonly used in the ordinary minimal string. There the marking procedure involves a derivative with respect to the boundary cosmological constant, which may seem contrived from the point of view of the worldsheet CFT.

With respect to the comment about BRST equivalence: it is not quite true that BRST equivalence is ascribed to the "timelike half" of the boundary condition alone, precisely because of the marking procedure. So it is not inconsistent that on the one hand the asymptotic boundaries created with different half-ZZ boundary conditions are equivalent while on the other there are genuinely different species of ZZ-instantons corresponding to each type of half-ZZ boundary condition. (In the case of ZZ-instantons, the BRST equivalence is ascribed to the spacelike sector of the worldsheet boundary condition, and is meant to motivate why we only consider $(1, 1)$ ZZ boundary conditions in the spacelike sector).

8. *The relation between the Virasoro minimal string (VMS) and Minimal Liouville gravity (MLG) could be exploited. Since $c \leq 1$ Liouville theory correlators are limits of minimal model correlators, it should be possible to deduce the VMS from MLG. The present article does not need this for its main claims. But when it comes to boundaries and instantons, things are on less solid ground: the half-ZZ branes are basically guesses, and invoking BRST equivalence without working it out is speculative. These issues are under better control in MLG. In particular, BRST cohomology has been studied in [30]. Is it possible to take limits of MLG in order to recover boundaries and instantons in the VMS? Presumably, one could recover either the asymptotic boundaries or the instantons, depending on how one takes the*

limit for boundary conditions.

We certainly agree that it is an important challenge to better understand the relation between VMS and minimal Liouville gravity. It is a nice suggestion to understand the relation between the two theories at the level of boundaries and instantons. Let us however mention a couple of conceptual points that may be obstacles to a full comparison between the two theories. First, in the (q,p) minimal string (with $q>2$) the dual description involves a multi-matrix model, and here we are interested in a limit that involves sending $p,q\to\infty$. On the other hand VMS is dual to a single-matrix integral. Also, the observables in the ordinary minimal string naturally live in the defect regime, corresponding to imaginary P_j in the language of the paper, whereas VMS is most naturally formulated in the regime where the external momenta are real. (However there is a window of values of the external momenta where the string diagrams may agree with the analytic continuation of the quantum volumes, see the discussion on page 84). In the $b\to 0$ limit these regimes correspond to the Weil-Petersson volumes with conical defects and geodesic boundaries, respectively.

9. *The quantum volumes $\mathsf{V}^{(b)}_{g,n}$ are reminiscent of the numbers of lattice points in moduli spaces of curves, as studied by Do and Norbury. Numbers of lattice points are functions of discrete parameters instead of P_i. Still, their expressions (in Appendix A of that work) are quite similar to those of $\mathsf{V}^{(b)}_{g,n}$. This might be related to the comment on Weil-Petersson volume on page 84.*

   We thank the referee for the interesting reference.

Comments about form

1. *"Solubility"*

   We have replaced this by "solvability".

2. *Figure 1: it is nice to have a road map, but it could be clearer and simpler. The relation between the worldsheet and the matrix integral is the article's main result. According to the beginning of Section 2.6, intersection theory is the bridge between the worldsheet and the matrix integral. But this is not clearly apparent in the figure, and the meaning of the arrows is not always clear. Moreover, boxes are sometimes redundant, with no hierarchy between important topics and minor points. The boxes about boundary conditions, large g asymptotics and properties of $\mathsf{V}^{(b)}_{g,n}$ might be superfluous.*

We would like to thank the referee for the encouraging comment. However we thought for a long time about the structure and connections we want to display in figure 1 and would rather not change it. The boxes to the secondary topics such as the large-g asymptotics and properties of the quantum volumes are meant to enable the interested reader to quickly navigate to the section of interest.

3. *The material in Section 3.2 could be moved to Section 8, since it is apparently not needed in Sections 4-7.*

   Similarly we think that section 3.2 belongs to section 3. This is in particular because we structured the paper in five different chapters and section 8 is part of "Evidence and applications." We think that it is appropriate to keep the description of the worldsheet boundary conditions in the section where we elucidate the worldsheet CFT.

4. *The claim in Section 2.5 that the matrix integral only captures primaries, and the stripping of the η function, deserve more explanations. The intuitive explanation is OK, but it is not clear how to make it technically more precise.*

   In principle the trumpet and disk partition functions could have included the contributions from the descendants via the eta function factors. This would have modified the resulting partition functions $Z^{(b)}_{g,n}$ in a trivial way. However we chose not to include them in the trumpet partition function in order to emphasize that the matrix model only captures the statistics of primaries.

5. *ψ_j is missing in the first line of (2.20)*

   We thank the referee for pointing out this typo. We have also fixed a similar typo in (4.14).

6. *The claim that (2.21) determines the quantum volumes at $g = 0, 1$ deserves more explanations.*

   We have added a reference to Witten's work on "Two-dimensional gravity and intersection theory on moduli space" (ref. [62]) where this is explained on page 16.

7. *(3.1) is a product of factors, not of terms*

   We have implemented the change from terms to factors.

8. *Page 24, the expression "gaps in the spectra" sounds imprecise.*

We have replaced “gaps in the spectra” with “the scaling dimensions of the lightest operators”.

9. *Page 28, the statement that conformal boundaries are “relatively weakly constrained” is vague. More precise statements would be welcome, such as which consistency constraints the half-ZZ branes obey and how this compares with the FZZT and ZZ branes.*

   What we meant by this comment was that in non-compact and non-unitary CFT, in implementing the cylinder bootstrap the spectrum in the open-string channel is a priori not subject to the usual constraints of positivity, discreteness, and integrality. Nevertheless the cylinder partition functions involving the half-ZZ boundary conditions obey these properties. We have added a footnote clarifying this on page 29. As for which consistency conditions we have verified for the half-ZZ branes, we have only checked the cylinder partition function (we have not checked any BCFT crossing equations involving boundary insertions).

10. *Page 33, the calculation (4.7) is a crucial step in the proof of the article's main result, and could be explained more carefully, by splitting (4.7) into several formulas interspersed with text. Moreover, dim Hg,n is infinite, and it comes back in (4.10). Both times it is stated that one could easily regularize this infinity. Would it be so tedious to actually do it?*

    We would rather not split this equation since we explain the intermediate steps in detail in the paragraph following eq. (4.7) and we believe that splitting it might lead to unecessary confusion. In this spirit we replaced the word “regulates” by “removes”.

11. *(4.12) and (4.14) reproduce (2.20) (minus the typo): this should be stated explicitly.*

    We have added this, thanks.

12. *Footnote 16 is probably superflous for any reader who survives that far.*

    We believe that footnote 16 is important, however to reduce the number of footnotes we happily moved it into the main text.

13. *In Section 4.5, the argument about degrees of forms that underlies the fact that the degree is $3g-3+n$ as well as equation (4.19) could be made more explicitly.*

    We have added a sentence above (4.18) clarifying this.

14. *The end of Section 5.2 is not that clear. I am not sure about the status of equation (5.14): is it announcing a result that will later be proved? Defining a new object that will turn out*

to coincide with the previously known object of the same name? Moreover, it is not clear how we deduce (5.13) from (5.12).

The leading density of states (5.12) and the more general resolvents in (5.13) are on slightly different footing (e.g. the Laplace transform of the disk partition function analogous to that in (5.13) does not converge). That the resolvents and the partition functions are related by inverse Laplace transform follows from their definitions in the matrix model. (5.14) then follows from inverting the fact that the partition functions are obtained from the volumes by gluing trumpets and (5.13). These are ultimately harmless operations because the quantum volumes are polynomials of finite degree.

15. *The triangle diagram Figure 6 is very helpful, but it could be made a bit clearer. To begin with it would be good to choose one notation $R^{(b)}_{g,n}$ or $\omega^{(b)}_{g,n}$. Then it is a bit confusing that two inverse Laplace transforms of the resolvent yield different results. It could be helpful to write in the diagram the relevant integral transforms in some concise notation (say for only one of the variables). Then the fact that the diagram commutes does not even seem to be stated although it is not trivial. As far as I can tell it boils down to an integral identity of the type (6.5) = (6.6), which could be stated and proved explicitly, as there could be subtleties with integration contours and absolute values.*

    We found it useful to have both representations of the resolvent. One has a more natural definition in the matrix model, but topological recursion is more naturally formulated in terms of the other. The two inverse Laplace transforms of the resolvents give different results because they are transforms of different variables. We included the hyperlinks to the equation numbers so that the reader could quickly access the precise equation without cluttering the diagram. Commutativity of the diagram at the level of fixed (g, n) is straightforward because the quantum volumes are finite-degree polynomials; this is not guaranteed for the resummed quantities that we discuss in section 6.

16. *The definition and role of S_0 could be clearer. Is (5.10) its definition? It would be nice to recall what is S_0 at the beginning of Section 6.*

    The use of S_0 is widely adopted in the literature on JT-gravity. We have added a reference to Saad-Shenker-Stanford just above equation (5.10).

17. *On page 50 it is announced that the ambiguities are the same as in Section 8.2. But it is not shown in detail why the (vaguely stated) ambiguities in deforming the integration contour*

correspond to the (rather precise) ambiguities of sign and branch cuts in 8.2.

We added more details in the caption of Figure 9.

18. *In the first sentence of 6.2, is it a typo to refer to the correction "to" $V_n^{(b)}(...)^{[1]}$?*

    Thanks. We have modified the first sentence to "From the leading non-perturbative correction $\mathsf{V}_n^{(b)}(...)^{[1]}$ to $\mathsf{V}_n^{(b)}(...)$...

19. *In Figure 10, it could be made clearer that these are functions of b.*

    It is unclear to us how we should stress the b dependence more than by having added superscripts on all quantities that depend on it. In particular the y-axis shows the quantum volume $\mathsf{V}_{g,0}^{(b)}/(\mathsf{V}_{g,0}^{(b)})_{\text{asym}}$ with explicit b dependency highlighted in the superscript.

20. *The definition of asymptotic boundaries is not clear to me. An asymptotic boundary is characterized by a renormalized length, which is computed as (8.4) for a particular type of boundary states, although it is not clear what it would be for a more generic boundary state. There is a statement that we need FZZT times ZZ-type (page 72), whose derivation (page 71) is hard to follow because it does not proceed from a precise definition of an asymptotic boundary. There is some tension between this statement, and the later summoning of BRST equivalence to say that things do not depend that much on the boundary state. Maybe FZZT times ZZ-type should be presented as an ansatz? It is called a "consistent prescription", and later "sensible conformal boundary conditions", but what do consistent and sensible mean in this context?*

    The (marked) FZZT times half-ZZ conformal boundary condition is essentially an ansatz, which we show correctly reproduces the matrix model results. In addition to the marking procedure that we have discussed in responding to a previous question, the main non-trivial step here is the identification of s^2 with a boundary energy. We find this natural because s labels the Liouville momentum of a state in the open-string Hilbert space. We do not know how to implement these boundary conditions at the level of the path integral and hence we lack the technology to derive this more directly in the sinh dilaton gravity theory. One may construct boundary conditions that correspond to the insertion of more general functions of the Hamiltonian in the matrix integral, but our claim is that this is the one that maps to the partition function of the matrix integral. One may view this identification as being fixed once one establishes the spectral curve via topological recursion and the disk partition function (e.g. from the relation to 3d gravity).

In view of this we have accordingly replaced “a consistent prescription” with “an ansatz that correctly reproduces matrix integral results.”

The later summoning of BRST equivalence is to argue that the precise species of half-ZZ boundary condition (when combined with FZZT boundaries in the spacelike theory and marked appropriately) is unimportant in describing the asymptotic boundaries, and similarly that the species of ZZ boundary condition (for the spacelike theory) is immaterial in describing the ZZ-instantons. When we say that the boundary conditions are “sensible,” we are referring to the fact that the cylinder partition functions are consistent with the interpretation in terms of a trace over the Hilbert space of the CFT on the strip.

We have accordingly added some more comments explaining the relation between the energy and the FZZT parameter in section 8.1