Referee 1

1. **Comment:** In Eq. (2.7), the authors show that the states belonging to the extended algebra are given by the tensor product of a state in the original algebra with a function g. This function g enters the final result of the generalised entropy in, e.g., Eq. (3.14) for the Rindler wedge and Eq. (3.21) for the ball-shaped region. This is analogous to the result of [Witten 2021], where the function g was determined to be a Gaussian for the thermofield double state. In this paper, however, the authors do not discuss how to determine g in the physical situations they consider. In particular, it is unclear if g is fixed or it remains free. The authors could comment on this function.

   Reply: The states relevant for the crossed product algebra live in an extended Hilbert space given by the tensor product $\mathcal{H} \otimes L^2(\mathbb{R})$, where the first factor is the QFT Hilbert space and the second is that of modular charge that is adjoined to the original algebra. This does not imply that all states on the extended algebra are tensor products. In Witten's original paper, the generalized entropy was derived for a *particular class* of states on this full Hilbert space, namely the ones that are simple tensors. Given that our construction is mathematical, we view g as a not necessarily fixed wavefunction determining quantum fluctuations in the modular Hamiltonian. We have commented on this in the revised version as per the referee's suggestion.

2. **Comment:** In Eq. (3.22) the authors consider the known result for the entanglement entropy of the vacuum of a 2d-CFT in an interval of length. The authors claim that the parameter a plays the role of an infrared cut-off. This may simply be a typo, but a is actually an ultraviolet cut-off on small lengths. A way to see that this is the case is the construction of [Cardy, Tonni 2016]. In that paper, $a << l$ appears as the diameter of two small circles that have to be removed from the boundary of the interval. This regularisation is needed precisely because the entanglement entropy is UV divergent. This is a minor point, but it is relevant to the stated purpose of comparing the known Eq. (3.22) with the authors' result in Eq. (3.21), since the generalised entropy (3.21) should be UV finite, differently from (3.22).

   Reply: We thank the referee for pointing this typo out. Of course we agree, a is indeed a UV cutoff necessary to regularize the entanglement entropy of the CFT, and the generalized entropy we obtain is UV finite without necessarily introducing such an ad-hoc cutoff. We have corrected this.

3. **Comment:**This last comment is optional, but i hope that the authors may clarify a doubt that arises with respect to excited states in AdS/CFT holography. At pgs. 24-25 the authors discuss the case of excited states in an holographic large-N CFT. They argue that by considering an excited state (keeping fixed the boundary region), the bulk dual region is modified in such a way as to include the bulk dual relative to the vacuum. As a consequence, the bulk algebra relative to the excited states is larger

and contains the one relative to the vacuum. My doubt regards the boundary algebra. Since the boundary subregion is kept fixed, one could naively expect that the boundary algebra does not change. Therefore it seems that the duality between boundary and bulk algebras does not hold anymore. I hope that the authors could clarify this point.

Reply: We thank the referee for raising this point: while the growth of the dual bulk region for a fixed boundary interval is standard in holography, it is indeed somewhat subtle within the present algebraic formulation. The resolution is as follows: when we speak of the boundary algebra of observables, we really mean the GNS representation of the abstract algebra in a particular state. Here, states are linear functionals on the abstract algebra, as opposed to vectors in a particular Hilbert space. The abstract algebra does not change when moving to an excited state, but the representation's properties may change depending on how different the two states are from each other. It may turn out that the representations are unitarily equivalent, but for truly excited states, we expect them to be distinct as depicted in our figure. We have added a footnote explaining this perspective.

Referee 2

1. **Comment:** *While any open spacetime region may be assigned a von Neumann algebra A(O), imposing further axioms on the assignment (1.1) leads to relations between the algebras defined on different regions* I find this statement misleading. We only associate C*-algebras to open sets in spacetime. I do not know of any ways to assign a von Neumann algebra to an arbitrary open set in spacetime without taking the double commutant that, in local QFT with timeslice axiom, enlarges the region to its causal completion. As the authors elaborate: For a topologically trivial region, in many quantum field theories, in the standard KMS state, one proves Haag's duality. Taking the double commutant results in von Neumann algebras that are associated with double cones, or causally complete regions. For example, see the "time-like tube theorem".

   Reply: We agree with the referee: any open region O may be associated a C^* algebra of observables $A(O)$, which may then be weakly completed to obtain a von Neumann algebra $\mathfrak{A}(O)$. The relations between $\mathfrak{A}(O)$ and von Neumann algebras localized in different (but possibly related) regions than O rely on what properties one requires of the net of algebras. For example, if one requires or has Haag duality, then $\mathfrak{A}(O'') = \mathfrak{A}(O)$ and the von Neumann algebra associated to O is indeed localized in a generically larger region O''. This was explained in the first paragraph, but we agree that the previous phrasing was confusing, and have now changed the problematic sentence to "...while any open spacetime region may be assigned a C*-algebra $\mathfrak{A}(O)$..."

2. **Comment:** *These infinities arise from the lack of a finite trace on the algebras of observables, thereby obstructing the existence of density states localized to the region. This is due to the fact that the algebra of operators in type III theories does not admit a*

tensor factorization, in contrast to type I factors appearing in quantum mechanics This statement is correct but misleading. Note that the issue of the existence of trace and tensor factorization are not to be confused. Type II1 algebras have finite trace but still do not admit a reducible presentation on B(H), required for tensor factorization.

Reply: We agree, and thank the referee for pointing out this misleading phrasing. We have fixed the wording to separate the two issues.

3. **Comment:** *In stark contrast to type III algebras, type II algebras do admit a well-defined trace, and hence a meaningful definition of von Neumann entropy for density states.* It is important to mention that the von Neumann entropy of type II algebras are still different in nature from type I entropy. It is crucial to remember that the type II infinity entropy is not derived from counting states, as opposed to the case of type I algebras.

   Reply: This is true. However, both are von Neumann entropies in the sense of being defined as $-\mathrm{tr}\rho \ln \rho$, and the referee's comment is about the interpretation of this formula, which is a different statement than the one we are making. Nonetheless, we have remarked on this in a new footnote.

4. **Comment:** *The issue with this in the type III1 algebra is two-fold: there is no finite trace with which to normalize the density states, and there is no operator in the algebra of the subregion that generates them* I suggest changing finite trace to trace. Whether the trace is finite or not is not relevant for the discussion. Note that the trace of type I infinity is not finite. I do not understand what is meant by there is no operator in the algebra of the subregion that generates them.

   Reply: We agree with the change to 'trace' instead of 'finite trace'. What we mean by the last statement is the following: it is a well-known result that the density state of a Rindler wedge is e^{-K} where K is the Lorentz boost generator. However, K is not an operator in the algebra of observables in the wedge. Taking the crossed product adjoins K in an appropriate way so that (roughly) e^{-K} makes sense as an element of the algebra. We have modified the sentence in the manuscript to clarify this.

5. **Comment:** *Physically, inner automorphisms are simply unitary transformations.* For clarity, I suggest adding unitary transformations that belong to the algebra. Because outer automorphisms are also unitary transformations. In fact, any automorphism of a von Neumann algebra can always be realized as a unitary transformation on the GNS Hilbert space.

   Reply: This is stated when introducing inner automorphisms in the immediately preceding sentence, but we have written it out as suggested for maximum clarity.

6. **Comment:** *It is a standard result in the theory of operator algebras that the crossed product of a type III1 algebra with an outer automorphism is a von Neumann algebra of*

type II. I do not think this is a correct statement. What the authors mean is the crossed product with the modular group. As a counter example, consider a theory with a finite group as global charge, e.g. Z2. The crossed product of the local algebra of two disjoint regions with the Z2 group of outer automorphisms corresponding to intertwiners is still type III1.

Reply: That is a typo and the referee is correct. Indeed, the crossed product with any compact group is not sufficient to change the type of a III_1 factor. We thank the referee for catching this.

7. **Comment:** *In general, we will take it to be the Hamiltonian of the commutant.* I do not understand what the authors means by the Hamiltonian for the commutant algebra. Note that if A is type III1 then the commutant is also type III1. Note that in the case of holography, the author of [27] had a definition of this operator. But that applies only to the holographic setups. This is one of my main criticisms of this work.

   Reply: This is simply ambiguous phrasing, which we thank the referee for pointing out. For all cases we consider in our work, the full modular Hamiltonian may be formally split into $H = K - K'$, where K is an integral of local operators in the region O and K' is an integral of local operators in the region O'. For example, for Rindler, K' is just the Lorentz boost acting in the opposite wedge. We have consistently called the "one-sided modular Hamiltonians" K, K' the "modular charges", precisely to avoid the confusion the referee raises, and have fixed this statement accordingly. We also point out that the definition of the subtracted modular charge in ref. [27] is purely formal, since it involves an expectation value in the type III theory.

8. **Comment:** Equation 3.8: I am not sure what this equation means. In the case of holography, there was a large N parameter and we had a boundary definition of this operator. Here, the use of GN and the equation 3.13 seem ad hoc to me.

   Reply: The first equation is just the standard vacuum subtraction scheme in any quantum field theory, regardless if there is a large coupling parameter or not. As mentioned in our previous comment, the expression (3.8), as well as the analogous large-N expression in the holographic context [27], are both purely formal: the presence of the large-N parameter does not circumvent this, as it addresses thermal fluctuations in the canonical ensemble rather than the UV divergence being formally subtracted. As for (3.13), this is a result that follows from the semiclassical Raychaudhuri equation, relating the modular charge of a subregion and the area of the entangling surface. This derivation incorporates backreaction which is where G_N comes from. The same result may be obtained by computing the conserved charge associated to boosts about an entangling surface. We have cited several different derivations of this relationship in the ensuing paragraph.

9. **Comment:** As far as I can understand the authors point out the observation that Haag's duality favors entanglement wedge over causal wedge. An observation that appears and has been discussed in the literature before; for example see 2008.04810. This is often interpreted as a non-uniqueness of the choice of von Neumann algebras one can associate to GFF algebras, for example see 2210.00013. There are correspondingly two modular flows, each corresponding to one choice of von Neumann algebras. I fail to see how the crossed product construction *"represents a refinement of Haag's assignment of nets of observable algebras to spacetime regions by providing a natural construction of a type II factor"* as claimed by the authors.

   The referee raises several points here; let us respond to them in turn:

   2008.04810: While this paper does mention Haag duality in the context of entanglement wedge reconstruction, it makes a different statement than the observation we are making here. In particular, it works in the so-called "code-subspace" framework, which corresponds to a truncated algebra of O(1) operator insertions. Moreover, the notion of a "reconstructable wedge" cited therein ultimately arises from working in a type I model. Collectively, these lead to further assumptions in order to obtain the switchover behaviour which are unnecessary in our approach. Nonetheless, it should certainly be cited as an early application of these ideas in the holographic context, and we thank the referee for bringing it to our attention.

   2210.0013: The non-uniqueness in this paper indeed refers to the choice of causal vs. entanglement wedge, but this is not an ambiguity of the holographic algebra since, as discussed in section 4 of our paper, the bulk dual of the boundary algebra cannot be the causal wedge. While one could consider the modular flow of the von Neumann algebra of the causal wedge, this is generically different from the relevant algebra under consideration here, which as we have explained is unique. We have however added a citation to this work for completeness.

   The refinement of Haag's assignment of nets of observable algebras: this appears to be unrelated to the previous two points, but is explained in equation (1.2) and the surrounding paragraph.