Reply to the report of the paper '*Precision magnetometry exploiting excited-state quantum phase transitions*'

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Dear editor and reviewer,

we would like to thank very much the reviewer first of all for her/his accurate and stimulating comments, and then also for the overall positive assessment of our results and of their importance. We have amended the manuscript according to all the reviewer's suggestions, and think that the paper in the present form is suitable for publication.

We now reply to the specific issues raised by the reviewer.

Reviewer:

I think the authors should also mention quantum sensing in systems showing a dynamical phase transition (for instance [1,2]), since those are also related to the ESQPT studied here.

Answer:

We have mentioned quantum sensing with dynamical phase transitions in the third paragraph of the introduction:

"The QFI also characterises classical and quantum phase transitions [6,17], dynamical quantum phase transitions [18,19], as well as phase transitions in steady states of dissipative dynamics [20-22], as it is proportional to the Bures metric in the state space (except for pathological, eliminable singularities [23,24]). Therefore, the QFI is expected to be much larger, i.e. superextensive, at critical points that separate macroscopically different phases, while it is at most extensive elsewhere. For certain topological [25] and non-equilibrium [20,21] phase transitions, the QFI can also be superextensive within an entire phase. All the aforementioned paradigms for phase transitions have then potential applications in precision metrology."

Reviewer:

Although I fully agree that the QFI here shows super-extensive scaling, I am not sure it couldn't be obtained otherwise.

...

To be clear, this doesn't affect the validity or interest of the results presented here; but I think the claims of the paper should be amended. These are threefold: the QFI allows to witness the presence of the ESQPT, we can design a protocol showing super-extensive scaling using excited-state preparation, and this super-extensive scaling comes from the ESQPT. It is this latter claim which I believe to *be incorrect. In this regard, there are multiple statements that should be rewritten or supressed: for instance "$\mathcal{F}_h$ exhibits a sharp peak close to the critical energy E_c, and its maximum value... increases with the system size N" and "The superextensivity of the QFI... is therefore a signature of the ESQPT" on p.7, "both $\langle E_k|S_z^2|E_k\rangle$ and $\langle \tilde{E}_k|S_z^2|\tilde{E}_k\rangle$ scale as $\mathcal{O}(N^2)$ at the critical point" on p.10, "their critical behavior is the key resource for enhanced precision" on p.19.*

Answer:

We thank the reviewer for raising an issue that points to the role of criticality for metrology As we understand the reviewer's arguments, we reworded several sentences in the paper in order to smooth our claims. We however think that the superextensivilty of the quantum Fisher information and thus performances of precision metrology are due to the excited-state quantum phase transition (ESQPT). After replying to the other issues, we provide informal arguments but not rigorous proofs. The new rewording conveys that the criticised claim is not a claim but our interpretation of the results; additional research could shed light on this conceptual issue.

We did not amend the sentence "$\mathcal{F}_h$ exhibits a sharp peak close to the critical energy E_c, and its maximum value... increases with the system size N" and similar sentences where we describe mathematical properties of some quantity evaluated when the field equals a critical value or when the energy equals the critical energy. Indeed, we think that these sentences just describe mathematical statements irrespective of the fact that precision metrology is due to the ESQPT. Accidentally or not, the QFI as a function of the energy is peaked at the value E_c which is also the critical energy, as shown in figure 4(a,b).

The sentence "both $\langle E_k|S_z^2|E_k\rangle$ and $\langle \tilde{E}_k|S_z^2|\tilde{E}_k\rangle$ scale as $\mathcal{O}(N^2)$ at the critical point" has been replaced with "The extensivity of the magnetisation also entails that $\langle E_k|S_z^2|E_k\rangle$ and $\langle \tilde{E}_k|S_z^2|\tilde{E}_k\rangle$ scale as $\mathcal{O}(N^2)$".

The sentence "The superextensivity of the QFI... is therefore a signature of the ESQPT..." has been replaced by "We suggest that the superextensive peaks of the QFI, occurring at magnetic fields equal to the critical values h_c^k and at energy equal to the critical energy E_c, is a signature of the ESQPT. From this superextensivity, we now estimate the scaling of energy gaps $|E_n - E_k|$ around the critical energy E_c, that is another feature of the ESQPT."

The sentence "their critical behavior is the key resource for enhanced precision" has been replaced with "our results lead us to suggest that the critical behaviour of Hamiltonian eigenstates is the key resource for enhanced precision".

Reviewer:

I found the way $\Sigma^*_{\mathcal{F}_h}$ was defined a bit confusing. From the caption of Fig.4, I take it it corresponds to the half-peak width of the QFI, but when expressed as a function of E/N? Could the authors write this down explicitely in the main text?

Answer:

We have corrected the text according to the reviewer's suggestion: "The width at half peak of the QFI expressed as a function of E/N" at page 7.

Reviewer:

More generally, I found the notations $\Sigma_{\mathcal{F}_h}(E_k)$ and $\Sigma^*_{\mathcal{F}_h}$ rather cumbersome, I would suggest something like $\Sigma_h(E_k)$ and $\Sigma_E(h)^*$ instead, to lighten up the notation and highlight which parameter one takes the width against.

Answer:

We have implemented the changes suggested by the reviewer. Consistently, we have replaced $D_{h_c}^k$ with D_k.

Reviewer:

After Eq.(8), there is a somewhat complicated argument to conclude that $\langle E_k|S_z|E_k\rangle$ *behaves like* N^κ*, with* $\kappa \sim 1.02$*. Instead, I would simply and immediately say* $\kappa = 1$*, since magnetization is an extensive quantity... To which one can add a quick comment stating that numerical analysis confirms this scaling (the 0.02 deviation is much more likely to come from finite-size effects or numerical errors in the implementations than from relevant physical effects, in my opinion).*

Answer:

We have implemented the change suggested by the reviewer. The new paragraph starts with

"The magnetisation is an extensive quantity, and thus $\langle E_k|S_z|E_k\rangle = \mathcal{O}(N)$. We have numerically checked this scaling for the minima of $\langle E_k|S_z|E_k\rangle$ attained at the critical points $E_k = \tilde{E}_k = E_c$ and $h = h_c^k$ (see figure 5(a)). The minimum values are fitted by $|\langle \hat{S}_z\rangle_{\text{min}}| = CN^\kappa$, resulting in $\kappa \simeq 1.02$ irrespective of the excited state $|E_k\rangle$ (see figure 5(b)), where the deviation of κ from 1 is due to numerical errors. The extensivity of the magnetisation also entails that $\langle E_k|S_z^2|E_k\rangle$ and $\langle \tilde{E}_k|S_z^2|\tilde{E}_k\rangle$ scale as $\mathcal{O}(N^2)$."

Reviewer:

After Eq.(12), it could be interesting to show $\rho(E)p(E)$ *and* $\mathcal{F}(E)$ *on the same figure, to illustrate how they overlap when* N *changes.*

Answer:

We plotted in figure 7 the functions $\mathcal{F}_h(E_k)$ and $\rho(E_k)p(E_k)$, with $\rho(E_k) = \sum_l \delta(E_k - E_l)$. The function $\rho(E_k)p(E_k)$ is plotted for the two choises $p(E_k) = e^{-\beta E_k}/\text{Tr}\, e^{-\beta H_h}$ with $\beta = 0.01$ and $\beta = 0.005$, and $p(E_k) = \left|\langle E_k|(|\downarrow_z\rangle)^{\otimes n}\right|^2$. Panel (a) of figure 7 shows the plots for $N = 800$, and panel (b) shows plots for $N = 1600$. The peak of $\rho(E_k)p(E_k)$ decreases with N. Indeed, $p(E_k) = e^{-\beta E_k}/\text{Tr}\, e^{-\beta H_h}$ scales as $1/N$ for large temperatures (small β), and $p(E_k) = \left|\langle E_k|(|\downarrow_z\rangle)^{\otimes n}\right|^2$ scales as $N^{-0.06}$ (see reference [52]). Nevertheless, the integrand of equation (12) remains highly peaked around the critical energy. Note that we have rescaled some curves in order to plot them in the same figure. We have added the following comment after equation (12):

"In order to visualise the overlap of the QFI $\mathcal{F}_h(E_k)$ with $\rho(E)\,p(E)$,these functions are plotted in figure 7, with two choices for $p(E_k)$. The first case is $p(E_k) = e^{-\beta E_k}/\sum_k e^{-\beta E_k}$ and corresponds to the state before the phase estimation algorithm being the Gibbs state at large temperatures. The second case, namely $p(E_k) = \left|\langle E_k|\big(|\downarrow\rangle\big)^{\otimes N}\right|^2$, occurs when the state before the phase estimation algorithm has all spins down in the z direction. Note that some functions are rescaled by suitable factors (see the legends and the caption) in order to plot all of them in the same figure. The peak of $\rho(E)\,p(E)$ decreases with increasing N, but the peak of $\mathcal{F}_h(E_k)$ increases with N so that $\rho(E)\,p(E)\,\mathcal{F}_h(E_k)$ remains highly peaked around the critical energy. When the initial state is $|\downarrow\rangle^{\otimes N}$, for instance, the probability to obtain the critical eigenstate decays very slowly with N, i.e. $p(E_k) = \left|\langle E_k|\big(|\downarrow\rangle\big)^{\otimes N}\right|^2 = \mathcal{O}(N^{-0.06})$ [52]. Therefore the averaged QFI scales as $\overline{\mathcal{F}_h} \sim N^{\gamma-0.06} \sim N^{2.01}$."

Reviewer:

"p.8: the notation $\hat{CU}_{2^j \Delta t}$ is fairly cumbersome, I would suggest something like $\hat{C}_U(j)$ instead".

Answer:

We have implemented the suggested changes, and consistently also at pages 11 and 13.

Reviewer:

p.9: "'For instance, psucc = 0.9 imples" → "implies"

p.13: "also the metrological performances are probabilistic" → "the metrological performances are also probabilistic".

p.15: "scale with N much slowly than the spacing" → "much more slowly"

still on p.15, "we show the robustness magnetometric performances" → "the robustness of the magnetometric"

p.15 still, "We then obtain superextesive QFI" → "superextensive"

Answer:

We have implemented the corrections.

We now reply in detail to the criticisms concerning the claim that the superextensivity of the QFI and of the sensitivity is due to the ESQPT. Since we only provide informal arguments, we agree to amend the claim in the paper, as the reviewer suggested.

Reviewer:

Even for $h = 0$, when we are seemingly far away from the transition, we still obtain super-extensive behavior. Hence, it seems to me that this N^2 scaling may not come from the ESQPT directly; rather, it comes from the fact that we pick up highly-excited states.

Answer:

ESQPTs are characterised by accumulation of eigenstates around the critical energy: the density of eigenstates shows a logarithmic divergence, see figure 1(b,c) and equation (2). This accumulation is responsible of the broadness of the QFI peak at the critical energy and at criticial values of the magnetic field. The fact that the QFI is maximised exactly at the critical field for each eigenstate is in our opinion a hint that ESQPT plays a role. The QFI decreases away from the critical field, and thus the best magnetometric sensitivity is achieved at the critical point h_c^k given the k-th eigenstate. As the QFI gives the best metrological sensitivity, we think that there exists some kind of connection between precision magnetometry, the superextensivity of the QFI, and ESQPT.

Moreover, the Hamiltonian is unitarily invariant under $h \to -h$, therefore the QFI of the k-th eigenstate in figure 2 shows also a symmetric peak at a negative field. The superextensivity of the QFI at $h = 0$ could also be a consequence of the decay away from the peak that is contrasted by the above symmetry requirement.

Finally, the metrological protocols we have proposed relies upon the use of the "critical eigenstate" corresponding to the maximum of the QFI. Although at $h = 0$ several highly excited eigenstates show superextensive QFI, it is not simple to project on one of these eigenstates on demand. So our ideal metrological source state are only the eigenstates at the critical energy.

Reviewer:

To be more precise, consider the two Hamiltonians S_x^2/N or S_z constituting the LMG. Taken individually, they do not display any ESQPT; yet their highly-excited states are Dicke states with high dipole moment, which can exhibit super-extensive sensitivity.

Answer:

The operators S_x^2/N and S_z commute respectively with σ_x^i and σ_z^i for all spin index $i = 1, \dots, N$. Therefore, eigenstates of either S_x^2/N or S_z can be product states but are highly degenerate, and Dicke states are permutation invariant states lying in each of the Hamiltonian eigenspaces. Moreover, these states (both the product and the Dicke ones) do not depend on external parameters and thus the QFI is zero (as the derivative of the density matrices with respect to the external parameter is zero). In order to use these states as probes in metrological protocols, one has to encode there information about an external parameter to be measured. For instance, these states can be injected into linear interferometers with the scope to measure the relative phase. Considering now this setting, product states do not exhibit superextensive QFI and sensitivity, while Dicke states do. Nevertheless, product states are simple to be prepared, while the preparation of Dicke states has a computational cost in terms of entangling operations. It is also difficult to implement optimal or nearly-optimal estimation procedures for interferometric setup fed with Dicke states, that might require Bayesian analysis and adaptive schemes. The Lipkin-Meshkov-Glick (LMG) Hamiltonian with the competition between its two terms has the advantage to encode the value of magnetic fields in its eigenstates with superextensive QFI. We therefore proposed metrological protocols exploiting this encoding, which are different than the above interferometric setup.

Reviewer:

As the authors state themselves on p.2, one would talk about a critical behavior when we have two phases with a "normal" extensive behavior, and a different, super-extensive behavior occuring only at the boundary. This is the case, for instance when we consider the ground-state of the LMG, for which the QFI shows $\mathcal{O}(N^{4/3})$ scaling at the critical point only, and $\mathcal{O}(N)$ elsewhere. Here, it seems to me we can find this super-extensive behavior everywhere.

Answer:

As the reviewer has outlined, the QFI is superextensive typically at critical points that separate different phases in most of the phase transitions. Nevertheless, there are phase transitions with superextensive QFI in an entire phase, like topological phase transitions with a gapless phase (Gu, Lin, *Europhys. Lett.* **87**, 10003 (2009)), or non-equilibrium quantum phase transitions (Banchi, Giorda, Zanardi, *Phys. Rev. E* **89**, 022102 (2014); Marzolino, Prosen, *Phys. Rev. B* **96**, 104402 (2017)). These phase transitions separate phases with macroscopically different features, and the QFI scaling is one of this features.

Moreover, in the aforementioned phase transitions, one considers the QFI of the "same" state (the ground state for topological phase transitions, or the steady state for non-equilibrium phase transitions) at different values of external parameters. In the ESQPT of the LMG model, we consider the QFI of "different" states (all the eigenstates) at different values of the field. If we fix the state, say the k-th Hamiltonian eigenstate, the QFI reaches its maximum at the corresponding critical value h_c^k and decreases away from it. This behaviour is similar to what happens to ground states in standard quantum phase transitions: the value h_c^k separates two macroscopically different features of the k-th eigenstate.

Reviewer:

If h is small enough (more precisely, for $h \ll N^{-2}$, then hS_z can be safely treated as a perturbation, and the eigenstates of the full LMG Hamiltonian are still given by $|m\rangle$, with energies m^2/N.

To compute the QFI, we need to consider terms of the form $\left(\frac{\langle n|S_z|m\rangle}{E_n-E_m}\right)^2$. Let us consider two neighboring highly-excited states, $|m\rangle$ and $|m+1\rangle$, with $m = kN$ and $k = \mathcal{O}(1)$. Then we have directly $\langle m|S_z|n\rangle = \mathcal{O}(N)$, and $E_{m+1} - E_m = \frac{(m+1)^2-m^2}{N} = (2m + 1/N) = \mathcal{O}(1)$. As the authors state shortly before Eq.(4) in the main text, these scalings are ultimately responsible for the superextensive scaling of the QFI. However, as we just showed, this behavior can be obtained for vanishingly small value of h, deep in the $h < 1$ phase, where the ESQPT should not play any role. This suggest that the behavior which is obtained here comes purely from the excited-state structure of S_x^2/N, rather than from the competition between S_x^2/N and S_z.

Answer:

The perturbative computation that the reviewer explained pertains properties of the full Hamiltonian $H_h = hS_z - S_x^2/N$. It gives the QFI of eigenstates of H_h for small but non-vanishing h. If h is identically zero, the derivative of the density matrix with respect to h vanishes and so does the QFI. Therefore, the superextensivity is due in our opinion to the competition between S_x^2/N and S_z.

Moreover, from the metrological perspective, the assumption $h \ll N^{-2}$ implies that the perturbative computation applies when the magnetic field is known to be small up to a precision much smaller that the estimation error given by the QFI, i.e. $\mathcal{F}^{-1/2} \sim N^{-1}$. Therefore, the reviewer's argument seems to be not relevant for metrological applications. Indeed, if we cannot approximate the Hamiltonian eigenstates with the eigenstates of S_x^2/N, the competition between S_x^2/N and S_z is not negligible.

We would also notice that it has been already shown for quantum phase transitions in the ground state (Cozzini, Giorda, Zanardi, *Phys. Rev. B* **75**, 014439 (2007)) and non-equilibrium quantum phase transitions (Banchi, Giorda, Zanardi, *Phys. Rev. E* **89**, 022102 (2014)) that the QFI can remain superextensive when a control parameter approaches zero, as in our case.

Reviewer:

Alternatively, we could also propose a protocol more in line with Ramsey interferometric protocol, in which we prepare the eigenstates of S_x^2/N, then let it evolve under a weak field S_z; in this case, we recover again a N^2 scaling, without relying on critical effects at any point. To conclude, it seems to me that the protocol discussed here may not really be called critical, in that it doesn't leverage the competition between two operators to give rise to a different behavior. Rather, its interest would lie in the preparation scheme, which allows to differ from usual approaches relying on one-axis twisting operations.

Answer:

We thank the reviewer for the interesting comparison. Firt of all, we remind that the eigenstates of S_x^2/N can be product states but are highly degenerate.

Product states do not exhibit superextensive QFI and sensitivity, but we can entangle degenerate product eigenstates to prepare an entangled eigenstate that is indeed useful for metrology. This preparation has a computational cost, and is not the simplest preparation of S_x^2/N eigenstates. The simplest way to prepare such eigenstates instead results in product states.

We agree that criticality does not play a role in the Ramsey scheme proposed by the reviewer. Nevertheless, we think that the competition between the operators S_x^2/N and S_z is indeed exploited in the scheme. If, for instance, one considers the field S_x instead of S_z, entangled eigenstates of S_x^2/N do not provide good metrological performances. It is the non-commutativity of S_x^2/N and S_z together with the initial entanglement that provide good metrological performances, and this non-commutativity is in our opinion a form of competition (as, e.g., the two operators cannot be diagonalised simultaneously). In the framework of phase transitions, the competition is implemented as competing Hamiltonian terms.

Moreover, we have exploited eigestates at critical energy for arbitrary fields $|h| < 1$, not only for small h, connecting the critical behaviour with the metrological performances of our specific protocols. On the other hand, one can implement the Ramsey protocol for precise measurements not only for small h but for any phase induced by the field S_z: the QFI for this protocol is $\Delta^2 S_z$ independent of the phase. The comparison with the Ramsey interferometry can then be extended also to standard quantum phase transitions, like that in the Ising model with transverse field, $H = -\sum_j \sigma_x^j \sigma_x^{j+1} - h \sum_{j=1}^N \sigma_z^j$ (for N even), that is critical for $h = 1$. One can feed a Ramsey interferometer with entangled ground states of $-\sum_j \sigma_x^j \sigma_x^{j+1}$, namely $\big(|+\rangle^{\otimes N} \pm |-\rangle^{\otimes N}\big)/\sqrt{s}$, which provide Heisenberg-limited precision in the Ramsey protocol even for $h = 1$. In our opinion, this protocol does not prove that the criticality of the quantum phase transition in the Ising model is not useful for metrology. On the contrary, we think that the Ramsey interferometry is alternative to critical metrology, and differently allocates metrological resources.

Yours sincerely,
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