Response to referees' comments

Dear Referees,

We appreciate your thoughtful and constructive feedback on our manuscript. Enclosed, please find a comprehensive response to the comments and suggestions you provided.

Sincerely,
Prashant Kumar, F. D. M. Haldane

Referee 1

We thank the referee for reviewing our manuscript and providing suggestions for improvement. Here is our response to the comments:

1. *"In particular, the 2/5 Jain state is not exactly the ground state of the V1 Hamiltonian in the lowest Landau level (a point that should be clarified in the text)"*

   We thank the referee for this suggestion. This has been clarified in the new version.

2. *The evaluations are done in the cylindrical geometry (suitable for MPS) with a finite circumference of the cylinder so the results are in the quasi-2D and not strict 2D limit.*

   We thank the referee for this comment. We have reached large circumferences L_y so that S_4, S_6 become nearly size independent in all cases. Moreover, for gapped FQH states considered here, the correlation lengths ξ are of the order of a few magnetic lengths, nearly an order of magnitude smaller than the largest L_y. Hence, there is no subtle difference between quasi-1D and 2D limits in our numerical evaluation of Eq. 29. A comment has been added in section V clarifying this point.

3. *One interesting case that could be worth discussing is that of the 2/3 Jain state which I suspect does not saturate the Haldane bound ...*

   We agree that it is very interesting to compute $\bar{S}_4, \bar{S}_6$ for Jain states in general. Unfortunately, we are not aware how to express them as matrix product states, which would make them accessible to the techniques utilized in this paper.

4. *Can the neutral excitations be used to probe the gapped/gaplessness of a quantum Hall state? For example, can one compute the S(k) of the Gaffnian or the PH-Pfaffian and show that it represents a gapless state?*

   Yes, this is an interesting possibility. If the neutral density mode (GMP mode) of a QH state is gapless at long wavelengths, then one can expect a non-analytic static structure factor at $k\ell \ll 1$. For example, it has been shown both analytically and numerically that in the composite-Fermi liquid at $\nu = 1/2$, $S(k) \propto (k\ell)^3$ in quasi-1D geometry (Science 352, 197 and Phys. Rev. B 106, 075116). It is believed to crossover to $(k\ell)^3 \log 1/k\ell$ in the 2D limit. Moreover, one can compute the "exact" first moment of the spectrum (such as in Phys. Rev. B 106, 115101) using single-mode "approximation" of GMP (Phys. Rev. B 33, 2481). As such, this technique can be considered a sufficient condition for the presence of gapless excitations in a QH state.

We also thank the referee for pointing out typos and suggesting a reference. We have incorporated these suggestions in the revised manuscript.

Referee 2

We thank the referee for reviewing our manuscript, providing suggestions for improvement and recommending publication. Here is our response to the comments:

1. "*The only exception studied is the nu=2/5 Jain state which, for an interaction consisting only of a positive V_1 pseudopotential, has a higher value of $\bar{S}_6$ than that predicted theoretically.*"

and
"*can the authors provide a physical picture or a hypothesis why the theoretical coefficient S_6 is an upper bound for the Laughlin and Moore-Read states while it seems a lower bound for the Jain states, at least the $\nu = 2/5$ state discussed here? Is that due to the projector to the lowest Landau level that needs to be taken into account in the Jain states?*"

We wish to clarify that even at $\nu = 2/5$, we observe $\bar{S}_6$ smaller than the theoretical prediction at the largest sizes when the orbital spin variance term is taken into account in Eqns. 25 and 26. Although in this case, the difference is small and the finite size oscillations are bigger. We have reworded the relevant subsection to reflect this.

2. "*The authors discuss the expansion of the static structure factor in very general terms such as to obtain a leading coefficient that is a rank-4 tensor. Because eventually only isotropic cases are discussed ...*"

   We thank the referee for pointing this out as a potentially confusing aspect of the previous manuscript. We have put emphasis on the general framework since, as shown in Refs. 2, 47 and appendix, a general statement of Haldane bound exists in anisotropic but translationally invariant and inversion symmetric FQH states. We do agree with the suggestion of orienting the reader that our numerical calculations only deal with the isotropic case. We have emphasized in several places including the introduction and section V that rotational invariance is a requirement for saturating the Haldane bound as shown in appendix B. Hence we focus only on the isotropic and at least weakly maximally chiral FQH states for numerical calculation, i.e. the ones that are not ruled out by appendix B and previous studies.

   We also thank the referee for pointing out the typo in Eq. 30.

3. "*In the same vein, the reader is confronted with a general discussion of the composite-boson picture of the FQH effect. However, only Laughlin states and composite-fermion (Jain) states are investigated numerically, along with the Moore-Read state. It might be useful to directly orient the discussion of Sec. III to composite fermions.*"

   While composite-fermions are very successful at explaining many fractional quantum Hall states, here we have used the composite-boson formulation since it is able to provide a separation between guiding center and cyclotron orbit degrees of freedom. When an FQH state lies purely in a LL, only the former contains information about the nontrivial correlations between electrons. This is demonstrated by the fact that guiding center spin is purely due to correlations within a Landau level and the properties of composite-bosons transform naturally under particle-hole (PH) symmetry, i.e. the guiding center spin and $\bar{c}$ are odd under PH. More recently, one of the authors has proposed (arXiv:2302.12472) that the quantum Hall phases should be thought of as electric quadrupole fluids where composite-boson is the underlying quadrupole. Several properties such as Hall conductivity, Hall viscosity, fractional quasi-particles can be understood using this perspective, especially when *continuous* rotational symmetry is absent. We have added a paragraph in the introduction section to motivate the composite-boson formulation.

4. "*on the contrary, to invest a bit more Sec. V where the main results are simply described. Here, the reader is missing a bit of discussion and interpretation, as I have mentioned above ...*"

   We thank the referee for suggestions. We have included a comparison with wavefunction overlaps in section V.A. Moreover, we have clarified that the reason for extra correlations in general FQH states is due to presence of both chiralities of gravitons and absence of strong maximal chirality in entanglement spectra in the discussion section.

We thank the referee for pointing out the missing definitions for some quantities in the manuscript. We have added them in the revised version.

Referee 3

We thank the referee for reviewing our manuscript, providing suggestions for improvement and recommending publication. Here is our response to the comments:

1. " *There's a lack of discussion regarding the experimental significance of the static structure factor, in particular how the coefficients may be measured.*"

   We thank the referee for this comment. We have included a discussion of the relevance of our calculation to experiments. In particular, the graviton spectra when measured using circularly polarized Raman scattering would contain both ± 2 chiralities. The numerical calculations of the kind presented in our paper can be used to compare with experiments by predicting the degree of graviton chirality, which may also aid the identification of topological order. This is demonstrated by a prediction that the ratio of spectral weights of the two graviton chiralities is ≈ 0.03 at $\nu = 1/3$ in the presence of Coulomb interactions.

2. " *It is not completely clear if Appendix A1 & B are new, or just rewriting [2].*"

   We thank the referee for pointing out a potential confusion. Appendix A1 is a rewriting of Ref. 2 to make the paper self-contained. Appendix B contains an explicit discussion of the conditions under which the bound is saturated, which is new. Moreover, we have added appendix C reinterpreting two of the spectral sum rules of Golkar et al in the guiding center approach used in the paper. The organizational paragraph has been modified to clarify this.

3. "*Being a numerical-driven paper using some DMRG package, there is a notable absent of discussion on the simulation parameters and their convergence criteria. Other numerical physics in the field should be able to reproduce the data.*"

   We have noted the bond dimensions, the most consequential simulation hyper-parameter, in Fig. 1. Additional details can be found in Refs. 15, 21 and 22.

4. "*The Gaussian envelope leads to a order $O(\ell^2/\xi^2)$ correction to many observables (e.g. ref 14), and it is natural to assume that this correction also appears in the structure factors. A proper error analysis from the effects of the Gaussian envelope should be performed. What is the convergence of the structure factors in terms of $1/\xi$?*"

   The value of $\xi = 6\ell$ chosen in this paper has been previously found to not affect results significantly. We have added a reference in the revised manuscript to clarify this.

5. "*The 2/5 Jain state admits a "model wavefunction" within the lowest and first LL (with projected delta-interaction). Maybe this model wavefunction is the closest one may get to a "maximal chiral" wavefunction. How do S4 and S6 compare to theory then?*"

   We thank the referee for a very interesting suggestion. It does appear accessible to our numerical approach and may elucidate the nature of FQH ground states that do not lie within one Landau level. Nevertheless, we believe it is out of the scope of our paper as it would necessarily involve multiple LLs. We leave a detailed analysis of this FQH state for future work.

6. "*I also found the section on "composite bosons" confusing. This is the section where c-, s and various quantities are defined, but these quantities exist beyond the composite boson picture. The authors should try to define these topological quantities in a more general context, and then discuss their properties in the Chern-Simons case.*"

   We thank the referee for this comment. We have added discussion motivating the usage of composite-boson formulation in the introduction section and other parts of the manuscript. Also, please see our response to Q3 of the second referee.

7. "*However, the behavior of all the quantities under PH is difficult to find in the current manuscript. Perhaps the authors can organize this information in a table.*"

   We have emphasized behavior of various quantities under PH symmetry in section IV. This should make it more accessible.