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I. RESPONSE TO REFEREE 2

We thank the Referees for taking the time to read our manuscript carefully and providing us with detailed reports. Below, we address the concerns raised and describe the changes introduced in the manuscript (all modifications, except for the additions in the bibliography, are marked in blue text). For the sake of convenience, we address the remarks of the Referee in quoting the full texts of the reports. For the same reason, we took a liberty to divide the report in parts, and answer them part by part.

- *The objective of the paper, analyzing how the topological pumping mechanism is modified by replacing the classical fields with quantized ones, feels relevant. The conclusions are backed up by a solid analysis. Nonetheless, personally, I find this paper very hard to read, not due to the complexity of the problem, but because I drown in technical details.*

  - We thank the referee for assessing the relevance of our work and for the valuable comment on the clarity of the presentation. As a consequence, we significantly simplified the technical aspects in order to emphasize the physical picture of the results. We moved the technical details in the appendices, and clarified in the main text the essential physical results of our approach.

- *The main conclusion that a qubit interacting with two quantized modes results in cat states is not very new, but rather general. In fact, two boson modes initially in coherent states and interacting with a qubit via a Jaynes-Cummings interaction will also result in an entangled cat.*

  - We thank the referee for the valuable comment. The remarkable aspect of the dynamics induced by a topological coupling is not only that any naturally prepared initial state splits into a cat state, but also that the two components of this cat separate in energy at a quantized rate and on arbitrary distance on phase space. Moreover, our full quantum treatment enables a quantitative description of (i) the entanglement between the qubit and the two quantum modes which was not provided before, (ii) the role of the quantum metric and of topological constraint on the nature of the adiabatic states, (iii) the time evolution of the quantum fluctuations of the modes' photon numbers and its relation with Bloch oscillations and Bloch breathing. None of these important properties of the dynamics of the model were discussed previously. In particular, we show that the entanglement between the qubit and the modes originates from the geometry of the coupling, which is characterized by the quantum metric. As a consequence, a topological coupling induces a high entanglement. This relation between topological coupling and high entanglement is a new discovery.

- *I lack a discussion about the relevance of the results. The light-matter interaction defined in eq. (2) involves the phase operators, and seems very particular. The meaning of phase states has a long history in quantum optics; both in terms of how to define and interpret a conjugate variable to the number operator, and how phase states can be prepared. Typically, the phase operators when expressed in terms of Fock states involve infinitely large Fock states, and I do not know how realistic the interaction of eq (2) is for real experiments. How could it be realized?*

  - We thank the referee for the relevant comment. Indeed, realistic light-matter interactions typically involve coupling between the Pauli matrices of a two-level system and the quadratures of a bosonic mode. We added the section 2.1.1 to define the regime of topological coupling between two harmonic oscillators and a qubit for such type of interaction. Our model of quantum rotors amounts to consider only the coupling between the qubit and the phase of the modes, while ignoring its coupling to the number of quanta of the modes. At the beginning of Sec. 2.1.2 We now discuss the conditions for this model to accurately describe the dynamics of the bosonic modes. The model of quantum rotors quantitatively describes the dynamics of states with quantum fluctuations of the number of quanta small compared to their average value . Furthermore, in Sec. 3.3 we explain that the phase states are indeed hard to prepare experimentally, as well as the adiabatic states (which are the only states which do not split into a cat state). Nonetheless, any initial states naturally decomposes of two adiabatic states. This is the reason why the dynamics leads generically to cat states.

- ► *The authors stress how the topological pumping with quantized fields leads to the build-up of entanglement between the qubit and the fields. Of course, this is a general statement and does not depend on topology. Indeed, others have asked similar questions in the past; what if we replace the classical field with a quantum one? The authors cite a few works already, and another one is Rev. Lett. 89, 220404 (2002).*

  - ▷ We thank the referee for the relevant comment. Indeed, the dynamics of coupled quantum degrees of freedom generically leads to an entanglement between them independently of the topological nature of the coupling. However, we unveil topological constraints on this entanglement. More precisely, we show that a topological coupling enhances entanglement: two bosonic modes are more strongly entangled with the qubit during the adiabatic dynamics for a topological coupling than for a non-topological coupling. Indeed, we relate this entanglement to the geometric properties of the adiabatic states (quantified by their quantum metric) which, as we show, are constrained by topology. We now make clearer in the manuscript this result on topology as an enhancer of entanglement.

- ► *The link to Bloch oscillations is mentioned. It brings to mind a recent work of Bloch oscillations in state space Rev. A 98, 053820 (2018).*

  - ▷ We thank the referee for this relevant reference. We added references in the beginning of section 4.2 on the manifestation of Bloch oscillations in other physical contexts involving an artificial lattice.