Strengths

1. An interesting qualitative prediction related with the formation of position dependent fluctuations of the order parameter after a quench.
2. Careful comparison of the initial transient behaviour to other approaches.·
3. Careful studying of the effects of disorder

Weaknesses

1. The chosen model for the adiabatic time evolution of the electron occupation number with the fast-changing temperature seems unphysical.
2. Many of the quantities in the numerics have been left undefined, making it difficult or impossible for others to repeat the findings (see report).
3. Use of dimensionless units is at some points confusing.
4. Manuscript does not provide information on the possible checks of how discretisation errors or finite-size effects affect the predictions.
5. Manuscript does not explicitly tell how the inhomogeneities are allowed to take place in the numerics in the clean case. In other words, how does the translation symmetry get spontaneously broken?

Report

This manuscript presents an interesting study of what happens in an ordinary two-dimensional superconductor when the temperature is suddenly brought from above the critical temperature to much below the critical temperature. Using a numerical solution of the Bogoliubov-de Gennes equation in a finite lattice, they show how the position averaged amplitude of the order parameter oscillates first in time, but after some initial transient the oscillations decay first in a power law and later exponentially. Then they show how the exponential decay of the oscillations is accompanied by formation of position dependent fluctuations in the order parameter amplitude. They also argue how the oscillations and the power law decay are consistent with earlier treatments of the problems within the position independent BCS approach, but the exponential decay and the formation of position dependent irregularities are naturally beyond them. All this is studied both in the clean and in the disordered limit, showing how the predicted behavior for the clean and weakly disordered cases are quite similar, and in particular the formation of position dependent irregularities does not require disorder.

The results of this study are interesting and I can probably recommend publishing them in SciPost physics after the authors have considered by comments. Namely, the authors do not completely specify what is done in their model (see below). Moreover, the results should be checked against possible issues in discretization. I also suggest them to discuss the results from two points of views, namely fluctuations in two-dimensional systems, and from the models used for the amplitude (Higgs) mode fluctuations in superconductors.

Our response:

We thank the referee for their time and effort in reviewing our manuscript and for the positive and constructive report. Some of the referee's requested changes have certainly led to a better manuscript. Below is a detailed response to the referee's comments and questions:

a) Model problems. There are some issues both in the assumptions and in the description of the employed model:

(i) Above Eq. (3), the authors state that the occupation number follows always adiabatically the time-dependent temperature within the fast quench. This seems unphysical to me, because in practice such adiabaticity requires very strong inelastic scattering (scattering rate exceeding the quench time scale, which on the other hand is comparable with the lattice energy scales). It seems to me that within this model the adiabatic quench is possible to reach only with the slow quench discussed in Appendix C.

Our response:

We thank the referee for the comment. Yes, we agree with the referee the adiabatic quench is not realistic for a sufficiently fast quench. However, in this paper, we focus on the pattern formation in space which typically occurs long after the quench protocol stops so the choice of quench protocol is not that important.

The paper results for different quench protocols (temperature and coupling constant quenches), and different quench speeds, indicate that the pattern formation in space is rather universal since, qualitatively at least, only depends on the lattice structure.

In the updated version of the manuscript, we have rewritten the text below Eq. (5) to make this point clearer.

(ii) The authors set the hopping constant t to unity, implying that all energies should be compared to t and time scales to hbar/t. Then, can the authors clarify if the tau_Q in Eq. (5) has a unit t^2/hbar, or is it perhaps rather related with T_c^2 as implied from the surrounding text.

Our response:

We note, in case the referee got confused with the notation, that in Eq. (5), “t” stands for time, not the hopping constant. We define the hopping constant as t_{i,i+\delta} in Eq.(2). Therefore, the time t has unit of hbar/t_{ i,i+\delta}. The temperature T has units of t_{ i,i+\delta }/k_B, where k_B is the Boltzmann constant.

Therefore, \tau_Q has units of t_{ i,i+\delta}^2 / (hbar *k_B). For numerical convenience, we set those constants as t_{ i,i+\delta}=1, \hbar=1 and k_B=1.

In the updated version of the paper, we make an explicit warning about this issue in order to avoid any confusion regarding the notation of time and the hopping constant.

(iii) Fig. 1 contains the symbol Delta_0 (vertical and horizontal scales), which later in the text is defined as the zero-temperature Delta. It would be better to define it on the first use and also provide its value in terms of t for the chosen parameters (is it 0.83 t as specified below Eq. (6)). Then, can the authors explain why the long-time limit of Delta is so much below Delta_0 (according to Fig. 4 this Delta \approx 0.83 Delta_0, is this a coincidence?). According to the BCS mean field model, Delta(0.1 T_C) is almost the same as Delta_0, the deviation is perhaps one permille.

Our response:
Thanks for the suggestion. In the updated version, we define it in Fig.1 as the referee suggests.
The fact that in Fig.4, \Delta\approx 0.83 \Delta_0 is just a coincidence since the prefactor depends on the coupling constant.

There is no reason to expect that the value of the order parameter for long times after the quench agrees with the static BCS prediction. For instance, in the weak coupling and long time limit, for the quench protocol specified in Ref. https://journals.aps.org/prb/abstract/10.1103/PhysRevB.71.134514, Delta(t=\infty) \approx 0.5 \Delta_0 due to the persistence of the incoherent collective motion of the Cooper pairs. Moreover, it is expected that during the time evolution after the quench, the temperature dependence of the steady order parameter \Delta is quite different from the BCS theory prediction. An explicit example of this feature is shown in Fig.2 of https://journals.aps.org/pra/abstract/10.1103/PhysRevA.73.033614 .
Moreover, the emergence of spatial patterns occurs due to the fact that superconductivity is suppressed, which makes the averaged order parameter even smaller.

(iv) The values of U and mu or at least Delta_0 (in units of t) should be provided in Fig. 1 as it depends on them.

Our response:
Thanks for the suggestion. We improved the caption of Fig.1 in the updated manuscript. The values of U, mu and Delta_0 are now provided in Fig.1.

(v) I did not find a statement about the boundary conditions employed in the numerics. They should be specified.

Our response:
We thank the referee for pointing out this oversight. In all cases studied in the paper, we employ periodic boundary conditions. This is explicitly stated in Section II of the updated manuscript.

(vi) In the clean case, nothing depends on position initially. Therefore, a position independent Delta must be a solution of the dynamics. However, this solution may be unstable to position dependent fluctuations which seems to be the case based on the results. In other words, in order to get a position dependence in the numerics, it has to be put in there either on purpose or accidentally. Can the authors please explain in the text how they did it.

Our response:
In the clean limit, we didn't put any inhomogeneity by hand in the initial state. The numerical error for the initial state is obtained from the solution of the BdG equations in the case of V=0 by full diagonalization. The numerical error for the calculation of the initial state is of the order of 10^{-16}, which is the highest numerical accuracy that we could reach. If we look at the Fig.5(a) the variance of the initial state is of the order of 10^{-28}, which means that the initial state already has a very weak spatial dependence, due to the numerical error, which is unavoidable. The finite time step in Runge-Kutta integration will also induces instabilities that contribute to the seed for the inhomogeneities. This is the seed for the large spatial inhomogeneities that form at much longer time scales. However, these spatial patterns that emerge from the out of equilibrium dynamics do not depend qualitatively on the seed. In other words, the mentioned seed of position dependent fluctuations is just one way to make the spatially homogeneous solutions of the BdG equations unstable towards the formation of the large spatial patterns in the order parameter.

In view of that, the referee may then ask whether our results are ultimately a numerical artifact due to the small numerical error. The answer to that is a resounding no. While in our numerical simulation the seed of spatial inhomogeneities is due to a finite numerical accuracy in experiments are a consequence of the combined effect of thermal and quantum fluctuations and the presence of imperfections or impurities in the sample.

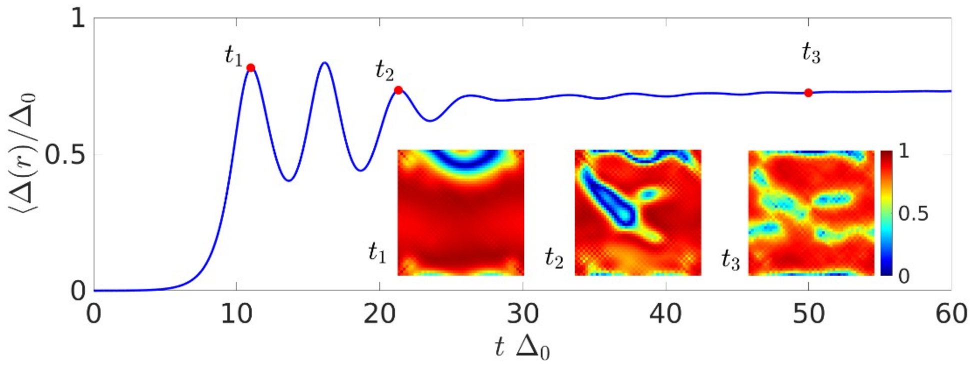

Figure R1. The dynamic of the clean system V=0. The initial state is completely random, which is not the self-consistent solution of the BdG equations. The system size is N = 48x48. The other parameters are the same with the paper. The coupling constant U=-3, the chemical potential \mu=-0.34.

We also did a check in a smaller system N=48x48 with completely random initial state, but still with no disorder V=0. The dynamic evolution is slightly different, but final equilibrium state is still qualitatively similar. See Figure R1 above, even though we obtain a curved strip which depends on the initial state at a much earlier time, but it gradually becomes the straight strips at long time evolution due to the underlying lattice structure. Indeed, the weak inhomogeneity or randomness in the initial state plays an important role in the dynamic system. However, in the clean limit system, the final equilibrium pattern should be qualitatively similar.

We have added an expanded explanation in the updated version.

(vii) In Sec. V, the authors claim that the length scale of irregularities is longer than the coherence length, but they do not tell how they define the coherence length, nor its precise value in their model.

Our response:
The referee is right, following the procedure of Ref. https://journals.aps.org/prb/abstract/10.1103/PhysRevB.92.064512 to extract the coherence length \xi_D from the fitting the intrinsic superconducting response \Delta D_s(q_y), we find that \xi_D \approx 1 in units of the lattice spacing. We have added this precise value to the updated manuscript in the last line of page 14.

(viii) The authors seem to claim that in all of their model, the phase of the order parameter remains essentially fixed. I find it surprising because a 2D system at a finite temperature should exhibit quite large phase fluctuations.

Our response:
Yes, we agree with the referee that in a 2D system at finite temperature, there should be large phase fluctuations. However, we are using a mean field model, which in fact do not consider the phase fluctuations, namely, the phase is fixed. Since we consider only the limit of weak or no disorder and the temperature is typically much lower than the critical temperature in the static limit, the phase fluctuation should play a less important role. Indeed, there is recent experimental evidence https://pubs.acs.org/doi/abs/10.1021/acs.nanolett.0c01288 supporting that this mean field BdG model, even without considering the phase of the order parameter, provides a valid description of the spatial inhomogeneities of a realistic weakly disordered superconductor. Going beyond this mean-field limit, by for instance including amplitude and phase fluctuations in the random phase approximation keeping the same system size, is technically challenging as it would require large computing resources both in RAM memory (well above 1T) and CPU (several hundreds of latest generation cores for sensible computation times) which is beyond our current capabilities. Another option is to simulate directly the attractive Hubbard model but, even with state of the art computing resources, the maximum sample size would be too small to reach any solid conclusion. In the updated manuscript, we briefly mention these technical difficulties in the introduction of the model section.

b) Possible issues in discretization

(i) The authors tell that 200x200 lattice is the largest that can be efficiently simulated. However, it would be important to understand the effect of discretization (or finite size effects) on the results. In particular, can the irregularities come from discretization? To check this, perhaps they could show what happens in a slightly smaller lattice (say, 150x150 sites): does the size scale of the irregularities change? If yes, can one argue what might happen in the thermodynamic limit?

Our response:

Yes, we calculated different system size and obtained similar patterns. However, we think it is not necessary to present all system sizes in the manuscript. Here we attach the results with the system size 120x120. The other parameters are the same with the paper. We obtain similar patterns in the long time evolution. As we have stated in the paper, our results points to a pattern formation that do not depend much on the details of the quench dynamics or the system size.

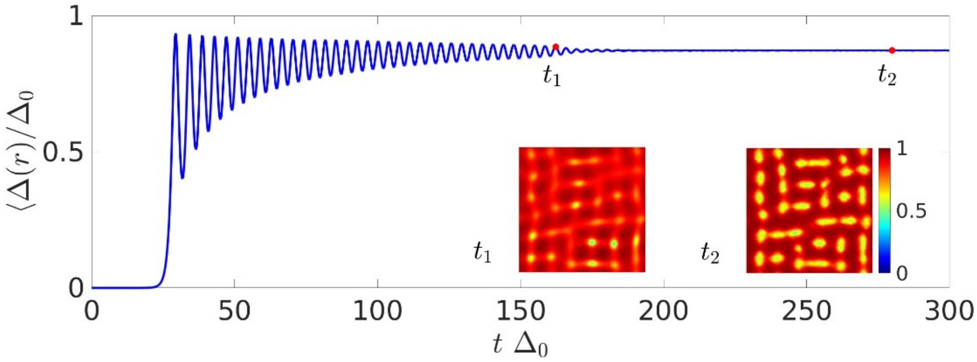

Figure R2. The dynamic of a smaller system. Except the system size is N = 120x120, the other parameters are the same with the paper. The coupling constant U=-3, the chemical potential \mu=-0.34.

(ii) The stripes in Figs. 7 and 8 are either horizontal or vertical. There are two possible reasons for this: they either orient along the lattice directions or along the edge directions, because the two are the same in the simulation. Which one is it? This should be possible to test by changing the relative directions of the edges with respect to the lattice directions (resulting into an irregular edge, but perhaps it does not matter).

Our response:

Thanks for raising this interesting issue. The referee intuition is indeed right. We have checked two different lattice structures and found that the stripes depend on it. In the updated manuscript, see Appendix B, we added new results for triangular lattice and obtain different stripes. We also studied its structure factor, which is a hexagonal pattern. The manuscript has been modified accordingly

c) Physical arguments: could the authors please discuss these

(i) We know from Mermin-Wagner theorem and Berezinskii-Kosterlitz-Thouless physics that in 2D there are no long-range correlations for the phase of the order parameter in superconductors. It is then unclear if there should be a related effect on the amplitude. However, it seems likely that the dimensionality matters for the fluctuations quite much. Can the authors at least emphasize that their results hold for two-dimensional systems, and perhaps speculate what might happen in three dimensions.

Our response:
The referee raises a valid point: there is no long-range order in two dimensions at finite temperature so, at least for sufficiently high temperature, fluctuations will be important. Since we are studying a mean field model, we are not able to consider the phase or amplitude fluctuations directly in our formalism. We note that if the temperature of the equilibrium state is sufficiently low we do not expect that our findings will not be affected much by BKT physics. Regarding the Mermin-Wagner theorem, it is well known that in typical experimental situations is in most cases avoided because the strict two dimensional limit required for its application is hard to reach for instance because of the coupling to a substrate. Therefore, at least for temperatures well below the BKT critical temperature, experimental observations are closer to the mean-field prediction which validates the use of the mean field limit in two dimensions.

Indeed, we expect that, in general, phase and amplitude fluctuations in two dimensions, which are more relevant than in higher dimensions, do not cause significant effects to the observed patterns. Indirect evidence of this is the fact that the quench dynamics starting with a random initial which can roughly simulate the complete absence of long range order, see Figure R1, leads to a similar spatial pattern for sufficiently long times. Moreover, our results in the weak disordered case also suggests that spatial inhomogeneities, similar to those caused by fluctuations, do not alter the emergence of spatial pattern for sufficiently long times. In fact, our results in the disordered case, especially when V=0.5, also support that this pattern against the effects of fluctuation and weak disorder. See Figure 8, when V=0.5, at earlier time t_2, disorder initially create some pattern structure that particularly depend on the disorder distribution or the initial state. But at long time evolution, it will gradually develop into a more ordered pattern structure that depends on the underlying lattice structure, see Figure 7.

Regarding what might happen in three dimensions, in this case, the mean-field limit that we employ is robust to BKT effects and the Mermin-Wagner theorem does not apply. Therefore, we expect that qualitatively similar patterns will also be observed in 3D. Results depicted in Figure R3 for the quench dynamics of a 3D superconducting system confirm this prediction. We observe similar oscillations in time in the early stages of the dynamics and the latter generation of spatial patterns so that the equilibrium state is spatially inhomogeneities. Our expectation is that for larger 3D systems, we would still obtain similar patterns that depend also on the lattice structure.

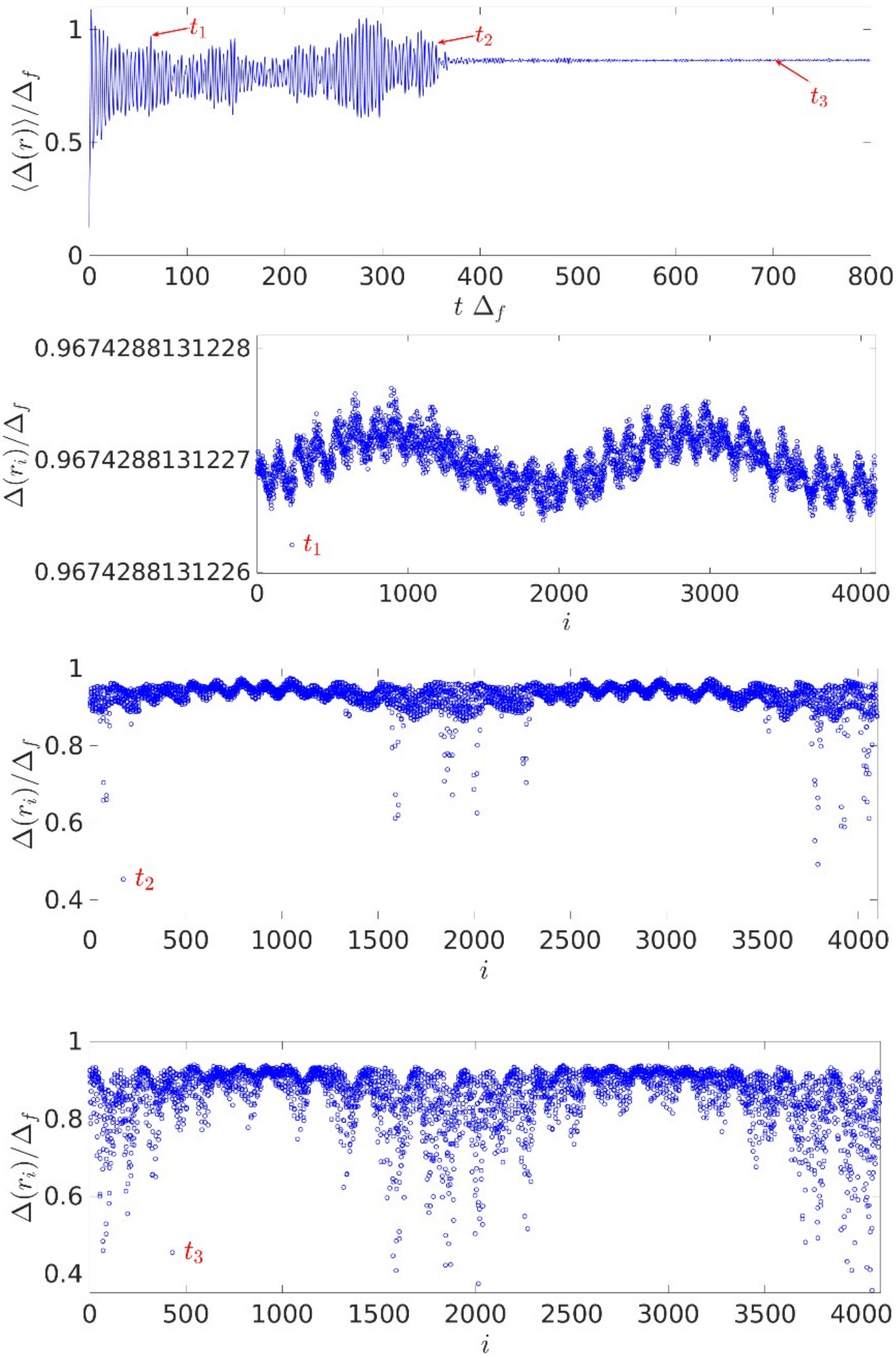

Figure R3. The dynamic of a 3D system with size N=16x16x16. The first plot is the time dependent of the spatial averaged order parameter, and the other three plots below is the corresponding spatial dependent order parameter at three representative time. We are simulating the sudden coupling quench. The initial coupling constant U_i = -2, and final coupling constant U_f = -5. \Delta_f \approx 1.73 is the BCS solution of the system with U_f = -5. The chemical potential \mu=0. For convenience, we plot the order parameter of each site \Delta(r_i) as a function of the label of the 3D sites.

(ii) Time-dependent fluctuations of the amplitude of the superconducting order parameter is closely linked with the dynamics of the amplitude or the Higgs mode. It behaves somewhat similarly to what has been discussed in here as it exhibits oscillations and a power-law decay of the order parameter (see, e.g., Eq. (1) in Moor, Volkov, Efetov, PRL 118, 047001 (2017)). Such models typically mostly consider the zero momentum Higgs mode. To my understanding, the results in the present manuscript imply that also the finite-momentum Higgs mode gets excited (as it describes the inhomogeneous state) and even stabilized.

Our response:
We believe, but we cannot demonstrate, that the referee intuition that the finite-momentum Higgs mode will get excited as a result of the quenched dynamics is correct. However, we would like to stress that the Higgs mode is an amplitude fluctuation whose study requires to go beyond the mean field approach that we employ in the paper. As we have commented previously, the study of the Higgs mode would require to compute corrections to the mean-field limit using for instance the random phase approximation or a direct calculation using the Hubbard model both of which are numerically demanding and beyond the scope of the paper.
In parallel, as mentioned earlier, we would like to stress that these fluctuations, that we do not consider, provide a source for the seed of the spatial inhomogeneities that in our mean-field formalism are due to the finite accuracy of the numerical results.

d) Small issues in the paper:
(i) Fig. 4 has the text "increases with disorder". Replace with "increases with increasing disorder"

Our response:
Thanks. We have fixed the typo in the updated version.

(ii) On page 9, the authors state "... as a function of time in the clean limit and in the absence of weak disorder". They probably mean "... in the presence of weak disorder".

Our response:
Yes, we mean “… in the presence of weak disorder”. We have fixed the typo in the latest manuscript.

(iii) Videos on Ref. 41 would require axis labels and some explanation of what the plotted quantities are. Perhaps you could include them in a single site along with a README text. Editors of SciPost might comment on the best way of storing them for long-term use.

Our response:
We have written a README text to explain the videos. We are also happy to follow the SciPost editors suggestion about storing the videos for long term use.

Requested changes

1. Define all employed quantities on their first use, and explain how the numerics is done (in particular, how it allows for spontaneous translation symmetry breaking)

Our response:

We thank the referee for this suggestion. We have substantially enlarged the introduction of the model and the details of the dynamic simulation so that with the provided information our results can be fully reproducible.

2. Check results against discretisation and lattice orientation

Our response:

We checked explicitly that a smaller system size (N=120x120, N=40x40 and N=100x100) leads to similar results so lattice discretization is not a problem. Moreover, we added the results for a triangular lattice system in the updated manuscript to demonstrate that the pattern formation does depend on the lattice structure.

3. Discuss results related to the BKT model and Higgs modes

Our response:

We have added discussions related to the BKT model, and the quantum fluctuations, which include the amplitude fluctuations (Higgs modes), in page 18.