Strengths

Well readable manuscript.

Weaknesses

Most ideas of this work are well known already in the community. In particular, works by Andrey Kolovsky based on his review from from 2002 and later papers, are the basic reference for Wannier-Stark problems without interaction. No interactions are treated here, then the problem is (basically) analytically accessible and less interesting, in particular for separable potentials that reduce always to one dimension only. Some issues must be fixed, see my report.

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Report

While the general setting is okay and the manuscript is well written, its contents lacks novelty. It is also no review-style paper since many relevant citations are missing. Most ideas of this work are well known already in the community. In particular, works by Andrey Kolovsky based on his review from from 2002 and later papers, are the basic reference for Wannier-Stark problems without interaction in one and more dimensions (being in separable or non separable Stark ladders). No interactions are treated here, then the problem is (basically) analytically accessible and less interesting, in particular for the separable potentials here studied that effectively reduce always to one dimension only. Some issues must be fixed:

We thank the referee for their expert comment. Here, we would like to provide further explanations regarding the motivation and to highlight the key aspects of our manuscript. In this work, we aim to investigate the impacts of both Hermitian and non-Hermitian Floquet terms on the dynamics of a Stark ladder system. We have obtained two main results:

(i) When the Floquet frequency is resonant with the slope of the linear field, the dynamics are peculiar. Theoretically, the dynamics are shown to be exactly non-periodic (or infinite long period). In contrast, it is periodic for situations at off-resonance.

(ii) At resonance, the spread of the probability distribution for the Floquet states exhibits two different types of behavior depending on whether the Floquet terms are Hermitian or non-Hermitian. To our knowledge, these results have not been reported in the literature. In the revised version, we will enhance the presentation of the manuscript and also make several corrections in response to the referee's comments. Our point-by-point responses are detailed below.

1) what is a Dirac probability distribution? Mathematically speaking this is well defined and not related to its use here, see around eq. (14-15).

2) For a non hermitian Hamiltonian, the left and right eigenstates are not necessarily orthonormal. This property is used throughout here, e.g. around eq. (15). The distinction between left and right eigenvectors should be made and proper motivations and definitions be given. What are the conditions for orthonormality?

We would like to respond to comments 1) and 2) with the following statement.

In the context of non-Hermitian systems, biorthonormal complete eigenstates refer to a set of eigenstates that are biorthonormal and form a complete basis for the Hilbert space of the system.

In a non-Hermitian system, the Hamiltonian H may not be Hermitian, i.e., it may not satisfy $H = H^{\dagger}$. It includes non-conservative terms that describe energy exchange with the environment. The eigenstates of a non-Hermitian Hamiltonian can be complex and may not be orthogonal to each other. However, if a non-Hermitian Hamiltonian has a complete set of biorthonormal eigenstates, it means that: There exists a left eigenstate $\langle\varphi_n|$ and a right eigenstate $|\psi_n\rangle$, satisfying

$$H|\psi_n\rangle = E_n|\psi_n\rangle, H^\dagger|\varphi_n\rangle = E_n^*|\varphi_n\rangle.$$

for each eigenvalue E_n such that

$$\langle\varphi_n|\psi_m\rangle = \delta_{nm}, \sum_n |\psi_n\rangle\langle\varphi_n| = 1.$$

The set of right eigenstates $\{|\psi_n\rangle\}$ and the set of left eigenstates $\{|\varphi_n\rangle\}$ together form a basis for the Hilbert space, allowing any state in the space to be expanded in terms of these eigenstates.

The concept of biorthonormal complete eigenstates is particularly important in non-Hermitian quantum mechanics because it plays an analogous role, both conceptually and computationally, to those in Hermitian quantum systems. For example, when we calculate the time evolution of a state, we first express the initial state $|\psi(0)\rangle$ as a linear combination of the right eigenstates

$$|\psi(0)\rangle = \sum_n |\psi_n\rangle\langle\varphi_n|\psi(0)\rangle$$

by projecting the initial state onto the left eigenstates. The state of the system at any later time t is given by

$$\begin{aligned} |\psi(t)\rangle &= \sum_n |\psi_n\rangle e^{-iE_n t}\langle\varphi_n|\psi(0)\rangle \\ &= U(t)|\psi(0)\rangle \end{aligned}$$

where $U(t)$ is time evolution operator, which also appears in eqs.(12) and (13) in the manuscript.

In parallel, we also have

$$\begin{aligned} |\varphi(t)\rangle &= \sum_n |\varphi_n\rangle e^{-iE_n^* t}\langle\psi_n|\varphi(0)\rangle \\ &= \widetilde{U}(t)|\varphi(0)\rangle. \end{aligned}$$

However, due to the non-Hermitian nature of H, the physical interpretation of a given state, particularly concerning probabilities and observables, may require additional considerations such as the use of a modified inner product.

For example, when we consider the probability distribution, the Dirac inner product is employed, i.e.,

$$P_l(t) = \langle\psi(t)|l\rangle\langle l|\psi(t)\rangle, \tag{1}$$

which is always real but $\sum_l P_l(t)$ is not conservative. In practice, the measurement of $P_l(t)$ has been reported in many experiments. In contrast, quantity $\langle\varphi(t)|n\rangle\langle n|\psi(t)\rangle$ may be not real but satisfies $\sum_l \langle\varphi(t)|n\rangle\langle n|\psi(t)\rangle = 1$.

3) P(t) is not at all necessarily period for real kappa, as stated in p. 5 in comparison to the figures! Please correct such statements.

According to the analytical analysis, P(t) is periodic with the period depending on ω and ω_0. When ω approaches ω_0, the period becomes infinite. The non-periodic behavior mentioned by the referee arises because the scale plotted in the figure is smaller than the period. We will emphasize this point in the revised version.

4) ref. 13 lacks two more authors, some ref.s are not complete with missing publishing company and place.

We will complete the revisions with additional information in the updated version.

5) a discussion is missing on how complex hopping can be realised in an experimental platform. Usually, some approximations are involved which should also be discussed. What does complex hopping do in the time evolution (conservation of probability, expected differences from real hopping case, ...), all this should be anticipated to prepare the reader for the results found here. I guess most of this has been studied before and a context setting should be given with appropriate references.

Thank the referee for their helpful and constructive suggestions. We will include the corresponding discussions and references in the revised version of the manuscript.

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Without a proper resetting of the citations and reference to the most important literature this manuscript should not be published.

Requested changes

Update of reference list, in particular add the many papers on the very topic of this submission, by Andrey Kolovsky and co-workers. Experimentally relevant papers by the Bloch and Goldman groups. See also the list 1)-5) of things to fix in the report.