First Report

The authors present a theoretical analysis of a three-terminal Josephson junction and evaluate critical current as a function of phase difference. They show with a simple theoretical analysis that a three-terminal Josephson junction with two ABS can realize JDE due to the relative phase shift between the ABS. I find the analysis and presented simulations for long and short junction limits to be of sound quality and agree with the conclusions. I think the paper can be published in this journal.

We thank the Referee for a detailed evaluation of our work and the positive feedback.

I have some minor comments and questions:

1. On page 6 in the description of Figure 4, I am assuming critical current as a function of ϕ_3 is evaluated by maximizing in ϕ_2 for a given ϕ_3 I think it will be helpful to state this clearly.

Indeed, this is how the critical current is calculated for this figure. We have clarified this point in the revised version of the manuscript.

2. To realize the proposed experimental setup, one would need to apply a magnetic field to phase bias a pair of terminals which breaks the time reversal symmetry. Can authors provide some comments on how their analysis would differ if the time reversal symmetry is broken?

The time-reversal symmetry-breaking effect of the magnetic field can be divided into two components: orbital effects, which cause phase shift of Andreev bound states [C. M. Moehle, et al., Nano Lett. 22, 8601 (2022)] (which ultimately leads to Fraunhofer pattern) and the Zeeman interaction, which causes splitting in the Andreev bound states spectra.

The maximal amplitude of the magnetic field required to operate the proposed three-terminal diode is such that it results in the 2π phase difference (ϕ) between the first and third terminal. When these two terminals are connected via a superconducting loop with radius R in the presence of the perpendicular field B the flux is $\Phi = \pi R^2 B$ and from the formula (neglecting the inductance of the superconducting loop)

$$\phi = \frac{2\pi\Phi}{\Phi_0}, \tag{1}$$

taking $R = 4207$ nm, we can estimate $B \approx 0.037$ mT.

To see if the orbital effect resulting from such a magnetic field would affect *the spectra of the considered junctions, we calculate the maximal ABS phase shift that would be experienced at the edge of the junction (with junction dimensions* $L = 500$ *nm and* $W = 500$ *nm) according to the formula,*

$$\phi^{'} = -2\pi \frac{fBLy}{\Phi_0}, \tag{2}$$

where $f = 6.2$ *is a typical focusing factor and* $y = 250$ *nm corresponds to the top edge of the junction. We obtain* $\phi^{'} \approx -0.0278\pi$ *indicating that the phase shifts due to the magnetic field are minimal and, therefore, the orbital effects can be neglected in our study. Note that this value has been obtained in the junction with dimensions exceeding the short-junction approximation, and for smaller junctions the shift would be even smaller. In fact, the magnetic-field-induced shifts are observed experimentally, but usually appear in much stronger fields (on the order of hundreds of microtesla). In this estimation, we used the typical circuit parameters from a recent experimental paper that probed ABS spectra in an external magnetic field by phase biasing a planar Josephson junction [C. M. Moehle, et al., Nano Lett. 22, 8601 (2022)].*

Using the estimated magnetic field, we can calculate the Zeeman splitting energy given by $E_z = g\mu_b B$, *where* $g = -51$ *is the Landé factor and* μ_b *is the Bohr magneton. With these values, we obtain* $E_z \approx 0.1$ μeV. *This energy is negligible compared to the gap of typical superconductors used recently in nanostructures (*$0.2 - 2$ meV *for Aluminum or Niobium superconductors), and therefore we expected insignificant changes in ABS spectra and supercurrents.*

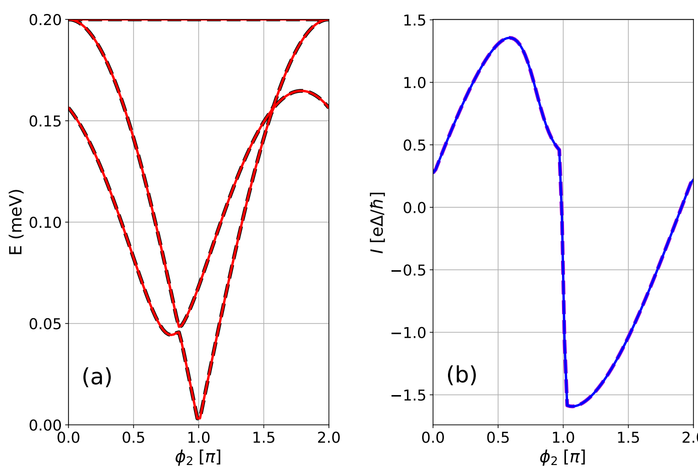

Figure 1: ABS spectra (a) and Supercurrent (b) with the effect of Zeeman interaction included (red and blue solid line respectively). Dashed lines represent the case without Zeeman effect

Finally, in Fig. 1, we compare the ABS spectrum (a) and the supercurrents (b) obtained without the Zeeman effect (dashed lines) and with the Zeeman ef-

fect included (solid lines) for the estimated magnetic field. We find no noticeable changes in the ABS spectra and supercurrents compared to those without the Zeeman effect compatible with the negligible value of the Zeeman splitting energy to the superconducting gap value.

We have added the appropriate comment in the manuscript.

3. In the absence of a phase loop, can authors provide some insight if the diode effect can still be realized in a three-terminal Josephson junction?

The diode effect can be realized without phase biasing in a tri-junction system (three Josephson junctions connected in a triangular loop) for the junctions that are characterized by a different critical current, leading to time-reversal symmetry breaking when the junction is biased by external currents. This has been experimentally demonstrated by J. Chiles, et al. in Nano Letters 23 (11), 5257 (2023). Alternatively, the field-free diode effect in multi-terminal junctions could potentially be realized by using inversion-breaking Van der Walls heterostructure $NbSe_2/Nb_3Br_8/NbSe_2$ *Josephson junctions, which so far were studied only in a two-terminal configuration, and which exhibited the diode effect even in the absence of external magnetic field [Wu et.al, Nature 604(7907), 653 (2022)]. We have added the appropriate comment in the manuscript.*

4. would like to point out a recent paper on three-terminal Josephson junction where JDE is observed along with π-supercurrent ($https://arxiv.org/abs/2312.17703$). Can authors comment if their model can account for this observation?

We thank the Referee for pointing out this important paper. The suggested manuscript [M. Gupta, et al., arXiv:2312.17703 (2023)] shows evidence of a π supercurrent contribution in a multiterminal system. In fact, in our model we do see π supercurrent components. This can be demonstrated by means of the analytical model (Eq. (3) of the manuscript).

In Figs. 2 (a) and (b) we show a situation analogous to Fig. 2 of the main text but with the phase on the third terminal set to zero. Now, upon phase biasing of the third lead we observe that the second mode (with Andreev bound state energy plotted in green) corresponds to the minimal free energy at $\phi_2 = \pi$ giving rise to the π supercurrent contribution (green in Fig. 2 (d)), which leads to overall modification of the total supercurrent in the system as compared to the $\phi_3 = 0$ case (confront the violet curves in Fig. 2(b) and (d)). We have added the appropriate comment in the manuscript.

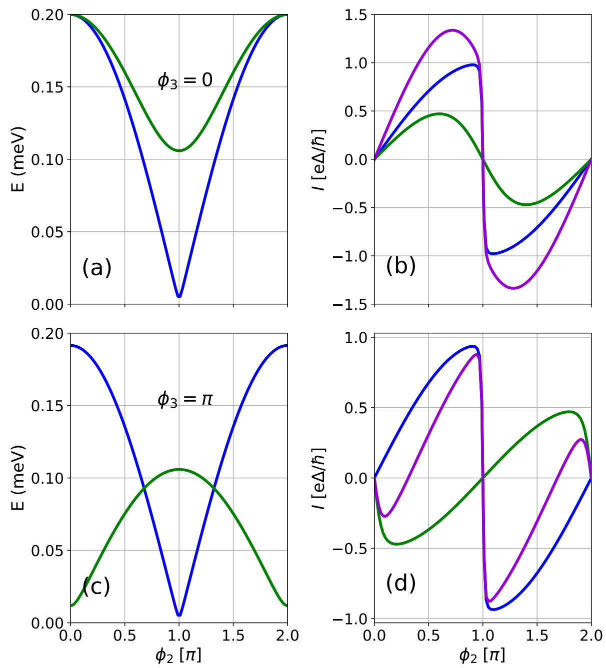

Figure 2: Top panels: energy spectrum (a) and supercurrents (b) in three-terminal Josephson hosting two ABSs for $\phi_3 = 0$. Bottom panels: energy spectrum (c) and supercurrents (d) for $\phi_3 = \pi$. The other parameters are the same as in Fig. 2 of the manuscript.