Second Report

Strengths

1 - The paper is nicely written and technically correct.
2 - Authors provide both analytical and numerical solutions that supports the claims in the paper and the paper contains a link to the code repository necessary for reproducing the results in the paper.

Weaknesses

1 - The idea of using a trijunction for a superconducting diode effect has been explored before.
2 - The paper does not provide a discussion on which regime (short- or long-junction regime) is better for implementing a superconducting diode.
3 - The paper lacks comparison with the previous three-terminal Josephson diodes reported in the literature.

Report

The manuscript explores the superconducting diode effect (SDE) in multi-terminal Josephson junction, focusing on a three-terminal setup. Traditionally, the SDE is associated with differences in critical currents for currents flowing in opposite directions in a two-terminal setup. However, the study reveals that in multi-terminal systems, this effect arises naturally without the need for spin interactions, arising instead from a relative shift between Andreev bound states (ABSs) carrying the supercurrent due to phase-biasing the third lead. The analytical findings for the short junction case is supported by numerical simulations and long junction case is investigated through numerical analysis. The results and findings of the paper is technically correct and the paper is nicely written. On the other hand, the idea of using multi-terminal Josephson junctions to break inversion symmetry and to break time-reversal symmetry by phase-biasing had been explored before. As authors cite in their paper, Ref. $[30-31]$ reports the superconducting diode effect.

The main novelty presented in the current manuscript is the use of a single scattering region. In my opinion, the difference is only incremental and the paper does not present a breakthrough or open a new pathway in the research area of the superconducting diode effect. Therefore, I believe the paper is more suitable for SciPost Phys. Core, and I recommend publishing there if the authors address the questions and comments.

We thank the Referee for careful assessment of our paper.

We do agree that there have been increasing efforts in realization of the Josephson effect in multiterminal systems, and the field is highly competitive. Indeed, recently there have been a few experimental papers that used multiterminal systems for the realization of the superconducting diode. In Ref. 30 (of the original submission) the diode effect was demonstrated in a multiterminal con-

figuration, but there the effect was based on the magnetic flux enclosed inside a three-junction loop. In fact in this and prior works [J. Chiles, et al., Nano Letters 23 (11), 5257 (2023)], due to extended system sizes and the lack of a small central scattering region, the two-terminal couplings are the dominant ones and the junction behavior is well captured by the electrical analog of a network of two-terminal Josephson junctions. The same model has been used to fit to the experimental results of Ref. 31 (of the original submission) which appeared just as we have been finishing the paper and which demonstrated that phase biasing can lead to the diode effect in a four-terminal Josephson junction.

The aim of our work was to provide a theoretical ground for the diode effect in multiterminal geometry with the most general, microscopic approach that would uncover the physical mechanism behind the diode effect. This is why our consideration is based on the single-scattering central region, without the need to resort to approximate circuit models. Our model captures all possible scattering events in the multiterminal Josephson junction—not only Cooper pair transfers, but also single-particle scattering between the superconductors (leading to crossed-Andreev reflections). We crucially demonstrate that the diode effect is a universal property of multiterminal junctions with broken rotational symmetry (which would otherwise result in a perfect beam splitting in the junction). We relate our results to previous studies that explain the diode effect in terms of time-reversal symmetry breaking. We believe that our work sets the theoretical background for future studies of superconducting diodes in multiterminal systems, where the number of modes and their couplings to the superconducting leads can be externally controlled and the three terminal processes cannot be neglected as recently demonstrated experimentally [M. Gupta et al., arXiv:2312.17703 (2023)]. This is why we believe that our work will open a new pathway in the promising topic of non-reciprocal superconducting elements.

Requested changes

1 - Can authors comment on the impact of the orbital effect due to the applied magnetic field and effects of disorder in the long junction limit?

The time-reversal symmetry-breaking effect of the magnetic field can be divided into two components: orbital effects, which cause phase shift of Andreev bound states [C. M. Moehle, et al., Nano Lett. 22, 8601 (2022)] (which ultimately leads to Fraunhofer pattern) and the Zeeman interaction, which causes splitting in the Andreev bound states spectra.

The maximal amplitude of the magnetic field required to operate the proposed three-terminal diode is such that it results in the 2π *phase difference* (ϕ) *between the first and third terminal. When these two terminals are connected via a superconducting loop with radius* R *in the presence of the perpendicular field* B *the flux is* $\Phi = \pi R^2 B$ *and from the formula (neglecting the inductance of the* *superconducting loop)*

$$\phi = \frac{2\pi\Phi}{\Phi_0}, \tag{1}$$

taking $R = 4207$ *nm, we can estimate* $B \approx 0.037$ *mT.*

To see if the orbital effect resulting from such a magnetic field would affect the spectra of the considered junctions, we calculate the maximal ABS phase shift that would be experienced at the edge of the junction (with junction dimensions $L = 500$ *nm and* $W = 500$ *nm) according to the formula,*

$$\phi^{'} = -2\pi\frac{fBLy}{\Phi_0}, \tag{2}$$

where $f = 6.2$ *is a typical focusing factor and* $y = 250$ *nm corresponds to the top edge of the junction. We obtain* $\phi^{'} \approx -0.0278\pi$ *indicating that the phase shifts due to the magnetic field are minimal and, therefore, the orbital effects can be neglected in our study. Note that this value has been obtained in the junction with dimensions exceeding the short-junction approximation, and for smaller junctions the shift would be even smaller. In fact, the magnetic-field-induced shifts are observed experimentally, but usually appear in much stronger fields (on the order of hundreds of microtesla). In this estimation, we used the typical circuit parameters from a recent experimental paper that probed ABS spectra in an external magnetic field by phase biasing a planar Josephson junction [C. M. Moehle, et al., Nano Lett. 22, 8601 (2022)].*

Using the estimated magnetic field, we can calculate the Zeeman splitting energy given by $E_z = g\mu_b B$*, where* $g = -51$ *is the Landé factor and* μ_b *is the Bohr magneton. With these values, we obtain* $E_z \approx 0.1$ μeV*. This energy is negligible compared to the gap of typical superconductors used recently in nanostructures (*$0.2 - 2$ *meV for Aluminum or Niobium superconductors), and therefore we expected insignificant changes in ABS spectra and supercurrents.*

Finally, in Fig. 1, we compare the ABS spectrum (a) and the supercurrents (b) obtained without the Zeeman effect (dashed lines) and with the Zeeman effect included (solid lines) for the estimated magnetic field. We find no noticeable changes in the ABS spectra and supercurrents compared to those without the Zeeman effect compatible with the negligible value of the Zeeman splitting energy to the superconducting gap value.

We have added the appropriate comment in the manuscript.

To assess the effect of the disorder, we performed an additional calculation for the junction beyond the short-junction approximation. The disorder is simulated as a random on-site potential with the value distributed uniformly between $-U_d/2$ and $U_d/2$, where the amplitude is given by:

$$U_d = \mu\sqrt{\frac{6\lambda_f^3}{\pi^3 a^2 l_e}}, \tag{3}$$

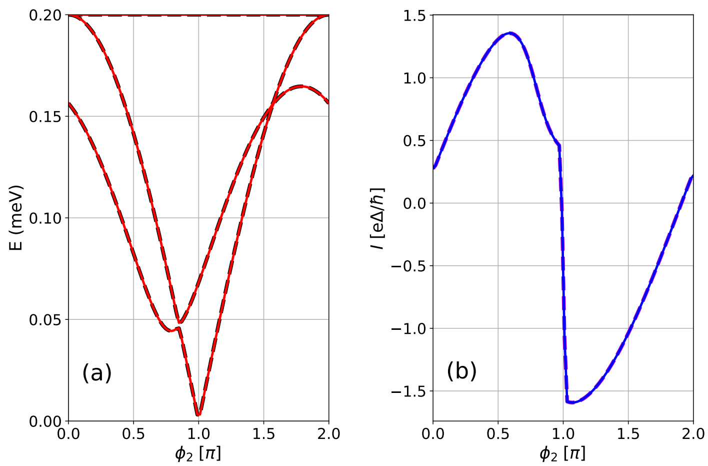

Figure 1: ABS spectra (a) and Supercurrent (b) with the effect of Zeeman interaction included (red and blue solid line respectively). Dashed lines represent the case without Zeeman effect

where μ is the chemical potential, λ_f is the Fermi wave length, l_e is the mean free path and a is the lattice constant [T. Ando, Phys. Rev. B 44, 8017 (1999)].

In Fig. 2, we present ABS spectra (left panels) and supercurrents (right panels) as a function of ϕ_2, considering three different mean free path values l_e with the rest of the parameters the same as in Fig. 11 of the manuscript. The disorder makes the junction opaque, resulting in an overall decrease in supercurrent. Moreover, when the mean free path is reduced, the ABS energy structure evolves into one composed mostly of flat bands, and the effects of phase shifts caused by external phase bias are less pronounced; hence, the diode efficiency is reduced. We find that for $l_e = 250$ nm the efficiency is $\eta \approx -0.63\%$, for $l_e = 500$ nm, $\eta \approx -1.77\%$ and for $l_e = 1000$ nm, $\eta \approx -5.11\%$.

In the revised version of the manuscript we added these new results together with the discussion of the effects induced by the disorder.

2 - How much do quasi-continuum states contribute to the diode effect in comparison to the subgap states?

The contribution of the quasi-continuum can be estimated by evaluating the efficiency for a varied number of junction eigenstates n (so the energy levels) that are taken into account when calculating the supercurrent. We estimate that for $n > 60$ the eigenvalues correspond to the ABS with energies exceeding the gap value $\Delta = 1$ meV and start to form the quasi-continuum [states above the red line in Fig. 3 (a)]. In Fig. 3 (b), we show the efficiency against the number of eigenvalues taken into account in the calculation of the supercurrent. Inclusion *states from the quasi-continuum (for n above 60) further decreases the efficiency up to the point around* $n = 300$ *when it finally saturates. The states from the quasi-continuum have only a slight phase dependence, but their abundance leads to a considerable contribution to the supercurrent and to the modification of the efficiency. We have added the appropriate comment in the manuscript.*

3 - A comparison between short and long junction regimes for the superconducting diode effect would enhance the impact of the paper.

Typically, for the junction to be in the short-junction regime, the dimensions of the junction have to be less than a few hundred nanometers (for the superconducting gap values of the order of fractions of milielectronovolt). In such junctions, there are only a few ABS due to transverse quantization of supercurrent-carrying modes. In Fig. 10 of the manuscript we show that as the size of the structure is increased, the efficiency drops as more and more conduction channels are populated. In fact, we see that the highest efficiencies are obtained for junctions populated only by a few modes (with the highest efficiency for junctions with length and width in the range of 100 - 200 nm, where there are only two transverse modes in the leads). On the other hand, the long-junctions, whose dimensions are comparable or exceed the coherence length, are naturally on the other side of the spectrum. The current is carried by many transverse modes, and the phase dependence of the continuum can also reduce the overall efficiency (see the previous answer). This is confirmed in our numerical calculations for the long junction, where we find efficiencies up to 20% *by fine-tuning the value of* ϕ_3*. Another important factor is that for a constant mean free path, the longer the junction becomes, the more the normal scattering dominates the ABS spectrum, further reducing the efficiency, again promoting short-junctions in terms of better efficiency. Therefore, the best efficiency will be obtained for small junctions, which fall into the short-junction category. This suggests that e.g. proximitized quasi-one-dimensional nanowires [Y. Cohen, et al., PNAS 115, 6991 (2018)] or nanoscopic 2DEG junctions [M. Gupta et al., arXiv:2312.17703 (2023)] are potential candidates for efficient superconducting diodes.*

We have added the appropriate comment in the manuscript.

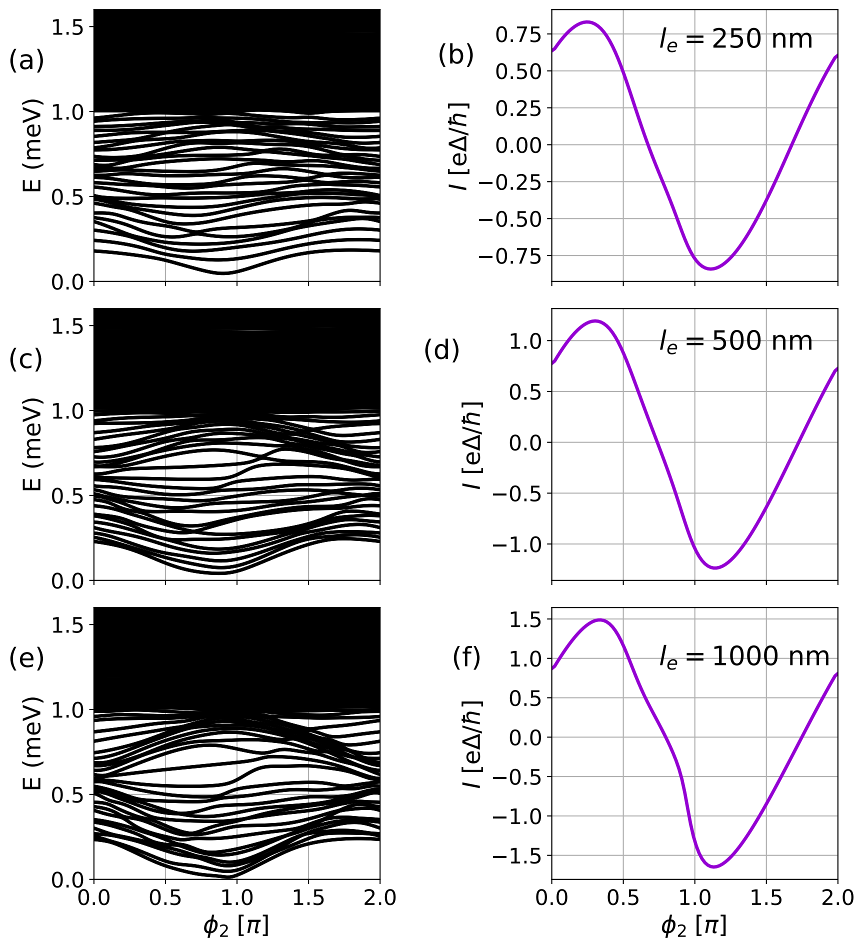

Figure 2: Andreev Bound states (left column) and supercurrents (right column) considering different mean free path values. $L = W = 500$ nm, $\phi_3 = 1.5\pi$.

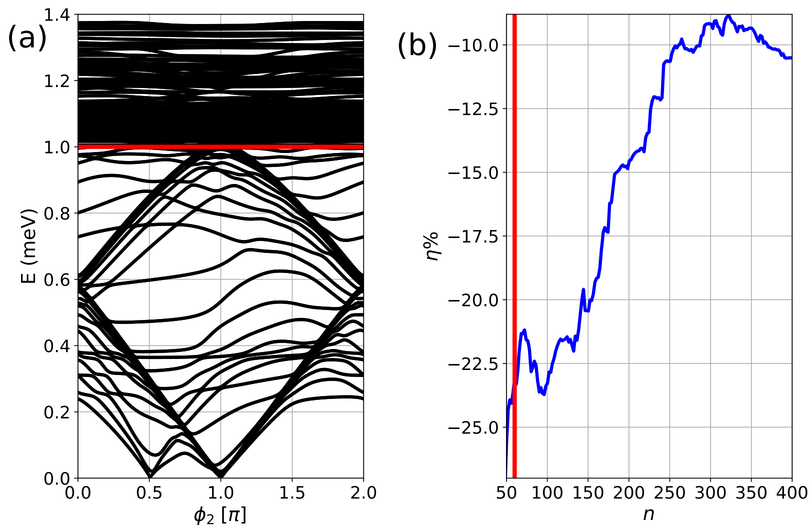

Figure 3: ABS spectrum of a long-junction including 400 eigenstates (a) and (b) the diode efficiency as a function of the number of eigenstates taken into account for the estimation of the supercurrent.