Response to both referees: summary of revisions

We are grateful to both Referees for their detailed and constructive reports that helped us improve our manuscript.

We have modified the paper significantly, incorporating important new material, and we have further highlighted the novelty and importance of our work for the field of superfluid vortex dynamics. Specifically, the current version of the manuscript contains the following new sections:

- Sec. 2: we demonstrate that the energy and dynamics of arbitrary vortex configurations can be easily calculated on complicated surfaces (whether curved or bounded) when one knows a conformal map that connects the complicated surface to a simpler domain, such as the entire plane or a circular boundary. This general unifying framework provides a much more compact derivation of our earlier results concerning the complex dynamical equation of a vortex with an elliptical boundary.

  We also prove that Hamilton's equations obtained from partial derivatives of the energy agree with our complex dynamical equations. Hamilton's equations conserve the total energy, which proves quite generally that the vortex orbits follow lines of constant energy. In particular, this result eliminates the need for separate proofs in each special case.

- Sec. 3: we summarize the behaviour of a vortex inside and outside a circular boundary, computing the equations of motion and the energy. This simple configuration will serve as a starting point to apply the general results in Sec. 2, with a circular boundary as the simple geometry and the Joukowsky map from a circle to an ellipse as the general shape.
- Sec. 5: we study a single vortex outside an elliptical boundary. This is a straightforward example of how the formalism of conformal maps, explained in Sec. 2, simplifies the calculation of vortex dynamical equations and energy. Streamlines and the phase field of the superfluid flow are shown in the new Fig. 1, while constant-energy contours are presented in the new Fig. 2.
- Appendix: we show that the dynamics of vortices inside an ellipse also follows from a very different method, based on a direct conformal map from a circle to the ellipse

Then, as compared to our first submission, we made the following revisions:

- Title: "in" is now replaced by "with" since we study the vortex dynamics outside of an elliptical boundary, too.
- Abstract: it now includes the general discussion in Sec. 2 and highlights the conservation of energy during vortex dynamics.
- Introduction and Conclusions: we updated them to account for the new material.
- we simplified the explanation of Joukowsky transformation, that was previously contained in Sec. 2, splitting it in two parts: in current Sec. 4 we give a general introduction, then we complete it at the beginning of Sec. 6. We also removed previous Fig. 1 and we improved previous Fig. 2 (that is current Fig. 3).
- In sec. 6 we study vortices inside an elliptical boundary. Compared to previous Sec. 3, we clarified the mathematical details. Exploiting the results presented in new Sec. 2, we simplified the derivation of both the equations of motion and the total energy. We removed the comparison with previous results expressed in elliptic coordinates to focus on our new treatment using complex variables.

  We made the original Fig. 3 more compact in what is now Fig. 4. Vortex trajectories in previous Fig. 4 are now combined with constant-energy contours in current Fig. 5. We added a legend in current Fig. 6 to repeat the color coding from current Fig. 5.

  Finally, we analyse a symmetric vortex dipole as a specific case of multivortex configuration, with new Fig. 8 showing the resulting streamlines and flow field, as well as constant-energy contours.

To our knowledge, earlier studies involving elliptical boundaries considered only the energy and other static properties. Here, we extend these works to include the dynamical behaviour of a vortex both outside and inside an elliptical boundary. Recent cold-atom experiments have studied real-time motion of a vortex in a circular boundary, and a similar study for an elliptical boundary would be of great interest.

In the following, we address all comments in the two reports in detail.

Reply to Referee #1

We are grateful to the Referee for their effort in assessing the manuscript, which helped us to improve its content. We answer in the following to all the points they raised.

<u>Report:</u> **This work studies the dynamics and energetics of a vortex in a two-dimensional superfluid confined within an elliptical boundary. The authors combine recent results for the energy of a vortex in an annular potential with a Joukowsky transformation to derive closed form expressions for the energy and equations of motion of a vortex in an elliptical container. Although an analytic expression for the energy has been derived previously, the newly presented derivation is argued to be simpler and more intuitive. The vortex trajectories resulting from the obtained expressions are presented, and it is found that they are slightly non-elliptical. The mean vortex orbital frequency is numerically measured, and is found to approach the known solution for a disk as the ellipse's aspect ratio approaches unity, as expected.**

While this work presents a novel derivation of the behaviour of a vortex inside an ellipse, I am not convinced that the manuscript satisfies the high acceptance criteria of SciPost Physics. This new derivation of a largely known result is not a groundbreaking discovery (criteria 1), nor a breakthrough on a long-standing research problem (criteria 2). It is not clear that it opens up new pathways for research (criteria 3), and I do not see how it links different research areas in a novel way (criteria 4). As such, I think this work is better suited for publication in a more specialised journal such as SciPost Physics Core.

In the new Sec. 2, we introduce a new and general formalism using conformal maps to obtain the dynamics and energy of a vortex with a quite general closed boundary. Earlier results discussed only the real static properties like the energy and streamlines. Our new complex formalism facilitates the study of vortex dynamics and shows that the trajectories agree precisely with those from Hamilton's equations based on the energy. We believe that our new complex formalism discussed in Sec. 2 definitely satisfies the criteria listed above.

<u>Weaknesses</u>

(1) Minimal new physics.

At the beginning of this response (in the introductory section common to both reports) we have highlighted how the revised version of the manuscript includes a substantial amount of new material (see in particular the new Sec. 2), which is broadly applicable to generic curved or bounded surfaces.

(2) Some mathematical arguments could be clarified.

We hope that the changes we made in response to the careful comments of both referees will have clarified the pending issues.

<u>Requested changes</u>

- **The authors refer throughout the manuscript to their newly derived expressions as "closed form", in contrast to the previous result of Ref. [17], which involved an infinite sum. However, the new expressions involve the Jacobi theta function, which itself is defined as an infinite summation. So:**
  1. **Is the new expression really "closed form", given that it involves infinite sums?**
  2. **Regardless of the semantics, how is it a significant simplification compared to the previous result?**

It is true that Jacobi theta functions are themselves defined as an infinite summation (or an infinite product), but so are familiar functions like $\sin z$. The theta functions can be easily handled by software like Mathematica and make evident the analytic structure of the complex potential through the zeros and poles. As noted in our revised text, this complex formalism also gives both the streamlines and the phase pattern, whereas the real formalism gives only the streamlines in the form of an infinite sum.

- **In paragraph 1 of the introduction, atomic gases are described as having "negligible compressibility". While the flow field may be incompressible if the atomic density is uniform, I would not describe the fluid itself as incompressible. It is a compressible fluid that can support both incompressible and compressible flows (ie. sound waves). Similarly, in paragraph 5, the authors state that a cold atom system in the Thomas-Fermi regime will support incompressible flow. However, again this is only true in a uniform system. As a counter-example, a BEC in a harmonic trap will have varying spatial density n in the Thomas-Fermi regime, and hence $\boldsymbol{\nabla} \cdot (n\boldsymbol{v}) \neq n\,(\boldsymbol{\nabla} \cdot \boldsymbol{v})$, meaning that the flow is not incompressible. I suggest the authors change the wording in these two places.**

We are grateful to the Referee for pointing out this subtlety. Following their suggestion, we modified the wording in both places. In particular, we specified that the incompressible flow is supported by uniform systems that are nowadays within easy experimental reach.

- **In the left frame of Fig. 2, I suggest labelling the angles $+\theta_0$ and $-\theta_0$ to emphasise that the angular positions of the two vortices are reflections about the X axis. It is also a bit confusing that the example configuration in the z-plane has the two vortices separated by ~ 90 degrees, since this is not the case in general. Using a different angle as the example might improve the clarity of the figure.**

For the sake of clarity, we have modified what is now Fig. 3 in the current version of the manuscript. In particular, we added vortex positions both inside the annulus (z_0, z_0^{-1}) and the ellipse (w_0), as well as the labels R^{-1}, R for the inner and outer edges of the annulus, respectively. Following the Referee's suggestion, we used a different angle for the coordinate z_0, in such a way that the two symmetric vortices inside the annulus are separated by more than 90 degrees. However, we have decided not to explicitly label the angles $+\theta_0$ and $-\theta_0$ because it would have made the left frame too dense of features, hence more confusing. We do think the meaning of the two angles can be easily understood by the caption and the discussion in the main text.

- **In the second sentence of Sec. 3, the authors quote from Ref. [33]: "in flow patterns related by a conformal map, circulation integrals around corresponding curves are the same". Looking at Fig. 2, I am confused by this. If one takes an integral around the orange path in the w-plane, the winding is $+2\pi$ since there is a single vortex enclosed. Likewise, taking an integral around the outer orange path in the z-plane gives $+2\pi$ (since this path encloses two positive vortices, plus a negative vortex at the origin of the z-plane, as described around Eq. 9). However, integrating around the inner orange path in the z-plane only encloses the central negative vortex, and hence the winding is -2π. Why do these windings around supposedly equivalent trajectories not agree?**

The symmetry $z \leftrightarrow z^{-1}$ implies a change in the sign of the polar angle $\theta \leftrightarrow -\theta$. Given two corresponding curves (for instance the orange circles mentioned by the Referee), this means that moving counterclockwise along the curve in the outer region is equivalent to moving clockwise along the curve in the inner region. The winding around the inner orange path is correctly -2π, performing the line integral in the counterclockwise direction. The symmetry of the Joukowsky transformation ensures that the inner orange path followed clockwise maps into the orange ellipse followed counterclockwise, explaining why the circulations along corresponding boundaries are consistently preserved.

- **In Sec. 3.1, the authors state that: "to ensure that the flow on the ellipse remains continuous across the branch cut, we must require $F_{\textbf{annulus}}\left(e^{i\theta}\right) = F_{\textbf{annulus}}\left(e^{-i\theta}\right)$ on the**

unit circle". It is not clear to me why this makes the flow continuous along the branch cut. Could the authors expand on this?

The Joukowsky transformation sends the unit circle $z = e^{i\theta}$ into the limiting ellipse with vanishingly small minor semiaxis b, i.e., the degenerate ellipse surrounding the focal line. In particular, the two points $z_1 = e^{i\theta}$ and $z_2 = e^{-i\theta}$ on the unit circle are sent to the same physical point $w = (z_i + z_i^{-1})/2 = \cos\theta$ along the focal line. For this reason, the complex potential $F(w)$ on the ellipse will be continuous across the focal line provided that $F_{\text{annulus}}\left(e^{i\theta}\right) = F_{\text{annulus}}\left(e^{-i\theta}\right)$. The corresponding material has been rewritten in this revised version and appears now in Sec. 6.1.

- **Does the argument presented in Eqs. (8) and (9) extend to the case of a multiquantum vortex in the ellipse? Presumably then one would need a negative multiquantum vortex in the center of the ellipse to ensure there is no winding around the unit circle?**

Indeed that argument does extend to the case of a multiquantum vortex inside the ellipse. To do so, one has to multiply current Eq. (31) by the charge Q of the vortex (the results we are presenting, in fact, are specific for $Q = +1$). The complex potential in the z plane, Eq. (33) in the present version of the manuscript, features a term $-Q \ln z$ that precisely accounts for a negative multiquantum vortex at the center of the annulus. The same final complex potential (35) in the w plane would then be multiplied by Q, as well.

- **Eq. (12) is the equation of motion for a vortex in the ellipse. Can the Joukowsky transform be used to map this velocity to the velocity of the equivalent vortex in the annulus?**

In the new version of the manuscript we have thoroughly addressed this point in Sec. 2.2. There, we derive the general relation between the vortex velocity on generic surfaces connected by a conformal map. Then, we apply it for the specific case of the Joukowsky transformation and we find the equations of motion for a vortex outside [Eq. (30)] and inside [Eq. (42)] an elliptical boundary. Of course, the general relation can be used to solve the inverse problem from the ellipse to the annulus. In this case, the conformal map to be used is the direct Joukowsky transformation $w(z)$.

- **Immediately after Eq. (12), the authors state that they "integrate the complex dynamical equation (12)". It would be useful to the reader to specify here that the integration is done numerically (as stated in the introduction).**

In the revised version of the manuscript we specify that the integration is done numerically, in the discussion below Eq. (42).

- **Regarding the results in Fig. 5, is there an intuitive reason why the trajectory is closest to being elliptical for the smallest orbital radius (blue curve)? Does the orbit become perfectly elliptical as the vortex approaches either the origin or the outer boundary of the ellipse?**

The parametric equation for an ellipse is $x = a\cos\theta$ and $y = b\sin\theta$. When recast into $x^2 + y^2 = \frac{1}{2}(a^2 + b^2) + \frac{1}{2}(a^2 - b^2)\cos 2\theta$, it clearly shows that an ellipse is a simple quadrupolar distortion of a circle with $\cos 2\theta$ dependence. Any deviation from an ellipse must involve higher-order even harmonic distortions like $\cos 4\theta$ with coefficients that depend on the radial variable. This behaviour is clear with elliptic coordinates, but we have chosen to omit this topic in favor of using our more powerful complex formalism with conformal maps (which we believe is new). These higher angular harmonics do become negligible near the center, ensuring that the actual vortex orbits are self-similar nested ellipses, as seen in our current Fig. 5. Near the outer elliptical boundary, the arguments used in connection with Fig. 2 ensure that the trajectories follow its elliptical shape.

- **In the final paragraph of Sec. 3.2, I suggest adding a citation for the quoted precession frequency of a vortex inside a disk.**

In the new version of the manuscript we derive it in Eq. (22) and we also add a reference to Lamb, *Hydrodynamics*, chap. 7 (current Ref. [5]).

- **Following Eq. (20), the authors assert that the vortex trajectories in Fig. 4 correspond to curves of constant energy. Could the authors plot these constant energy curves calculated from (20) in the figure, for comparison with the numerically integrated trajectories?**

Current Fig. 5 provides a direct comparison between curves of constant energy (left frame) and numerically integrated trajectories (right frame). As we discuss in the main text, the two results agree as expected from the equivalence of Hamilton equations with the equations of motion derived through the analytic properties of conformal maps. We have explicitly commented about the relation between iso-energy contours and vortex trajectories at the end of Sec. 2.2.1.

Reply to Referee #2

We are grateful to the Referee for their effort in assessing the manuscript, which helped us to improve its content. We answer in the following to all the points they raised.

Report: The manuscript investigates the dynamics of a point vortex inside a bounded elliptical domain. The key result of this work is that the authors use Joukowsky's conformal transformation to map an annulus domain into an elliptical one whereby a simpler problem can be solved. This yields an alternative representation of the point vortex dynamics in an elliptical domain that the authors allude to being more concise than previous methods using infinite images. Moreover, the authors show that the resulting vortex evolution is of a form of a near elliptical orbit that exhibits a small periodic deviation, which are then investigated across the parameter ranges of the elliptical domain.

The manuscript is extremely well-written and is both clear and concise (general criteria 1). The work is relevant to a wide audience, especially those associated with vortex dynamics in fluid and turbulent flow of both classical and quantum fluids, where the use of point vortices as an approximation is frequently used. The results are interesting and provide new insight to those studying vortex dynamics in elliptical domains, such as those, alluded to by the authors, working in condensed matter BECs, where quasi-2D quantum turbulence is now a predominant focus in the community with the application of vertically confined traps. However, I struggle to see the significance of this work. It is not totally clear to me in what way their analytical results are simpler to previous results for the same problem. Moreover, as the study is for only one vortex only, do their results extend to any multi-vortex configuration? This means that I find it hard to attribute one of the expectation criteria for publication is SciPost Physics.

I am not confident that publication in SciPost Physics is warranted due to the lack if significance, breakthrough, or novelty of their results. Alternatively, a more niche-focussed journal may be more appropriate.

Weaknesses

(1) Lack of significance of results.

(2) Reformulates pre-existing results via a new method.

(3) Not clear on the generality of the final results.

We responded in detail to these issues in the introductory part of our response (aimed to both referees). In our view, the revised version of our manuscript contains several new and important results concerning vortex dynamics in the presence of a general closed boundary or curved surface, which we consider significant and of general interest to a broad audience.

Requested changes

(1) Can the authors explain why their results are simpler than previous works in this setup? Is it the compact form of the analytical expressions or there is some computational element?

The compact form yielded by our complex formalism is the first element of simplicity, which leads to a straightforward numerical computation of relevant results like the stream function and the vortex dynamics. As an additional bonus, without extra effort the complex formalism gives additionally access to the phase field. An additional clear example of the simplicity of our new formalism is illustrated in Sec. 2.4.

(2) Point vortex simulations can be notoriously finicky with regards to numerical convergence (particularly when simulating vast multi-vortex configurations). I think it would be prudent if the authors included some technical details on the specific numerical methods they use.

We agree with the referee that computing the trajectories of a large multivortex array is a delicate numerical task. On the other hand, simulating the dynamics of a single vortex is a much simpler problem, requiring only the solution of two coupled first-order differential equations. We solved this problem with standard numerical methods (more in detail, using Mathematica's routine NDSolve) and we have verified that throughout all our simulations the energies were conserved to better than a part in a million.

(3) Subsequently, the authors compute a formula for the total energy of the point vortex system in the elliptical domain and assure the reader that it is conserved. It seems prudent to ask the authors to compute this energy during their simulations to ensure that it is indeed conserved and to what degree?

Almost by definition of ideal hydrodynamics with no dissipation, we believe that the energy is conserved. We now prove in Sec. 2 that the real and imaginary parts of the complex equation for the vortex dynamics are exactly the same as Hamilton's equations obtained with the vortex energy. Hence there is no need to verify this result in particular cases. In our view, this is one of the new and important results in our revised version.

(4) How does your results extend to multi-vortex configurations? Particularly to the argument of using equations (8) and (9).

In the new Sec. 6.4 we explain how to generalize the argument to multi-vortex configurations, providing results for the specific case of a symmetric vortex dipole.

(5) In Fig. 5, the amplitude of the periodic deviation decreases with the initial position of the vortex to the centre of the ellipse. Is there an explanation for this? How is it related to the position of x_0?

As discussed in our detailed response to the first referee, an ellipse is a simple quadrupolar distortion of a circle with $\cos 2\theta$ dependence. Any deviation from an ellipse must involve higher-order even harmonic distortions like $\cos 4\theta$ with coefficients that depend on the radial variable. This behaviour is clear with elliptic coordinates, but we have chosen to omit this topic in favor of our more powerful complex formalism using conformal maps. A detailed analysis for a vortex near the center shows that the contribution of higher Fourier harmonics like $\cos 4\theta$ is small, so that the shape is close to purely elliptical. Near the outer boundary, the arguments used in connection with Fig. 2 ensure that the trajectories follow the elliptical boundary shape. Figure 5 makes this behaviour clear.

(6) I suggest that Fig. 5 should include a legend to help explain the different colour lines rather than referring to the legend of Fig. 4.

We have added a legend in the figure, that is now labeled Fig. 6 in the current version of the manuscript.

(7) The sentence just before equation (13) needs rewriting as I believe that the grammar is not correct.

This specific sentence does not appear in the revised version of the manuscript.