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I. RESPONSE TO REFEREES

We thank the referees for taking the time to read our manuscript carefully and providing us with detailed reports. Below, we address the concerns raised and describe the changes introduced in the manuscript (all modifications, except for the additions in the bibliography, are marked in blue text). For the sake of convenience, we address the remarks of the Referee in quoting the full texts of the reports. For the same reason, we took a liberty to divide the report in parts, and answer them part by part.

A. Response to Referee 1

- *The authors discuss the significance of exact or approximate phase states, and I have the following thought. Rather than using canonical phase and amplitude variables, we could instead consider quadrature operators x and p. This substitution would modify the Bloch Hamiltonian in equations (5) to (up to commutation relations):* $h_x = \Delta x/\sqrt{n_x}, \quad h_y = \Delta y/\sqrt{n_y}, \quad h_z = \Delta(2 - p_x/\sqrt{n_x} - p_y/\sqrt{n_y})/4.$ *By ignoring fluctuations in n_x and n_y, this simplifies the model significantly, making it easier to realize experimentally.*

  - A coupling between the two-level system and the quadratures of the modes is indeed experimentally more natural than a coupling to the phase variable. This is why we use the quadrature variables in Sec. 2.1.1 to define the notion of topological coupling between the modes and the two-level system. We then introduce the model of quantum rotor in Sec. 2.1.2 as an approximation of the first model. This approximation is valid for initial states with small fluctuations in n_1 and n_2 and on short timescales as explained in the text. On longer timescales, the coupling to the quadratures leads indeed to interesting rich physics which will be studied in a separated work.

- *Moreover, a similar model, with $h_z = \Delta/2$, has been previously studied and it directly follows how it can be used for generating cat states (see Phys. Rev. A 81, 051803(R) (2010)). This simplified model has also been recently realized experimentally (see Nature Chemistry 15, 1509 (2023) and Nature Chemistry 15, 1503 (2023)). My question is: To what extent do the present results hold in these simpler models? For example, are the $\cos(\phi_1)$ and $\cos(\phi_2)$ terms in h_z essential?*

  - We thank the referee for the relevent references. The $\cos(\phi_1)$ and $\cos(\phi_2)$ terms in h_z are essential to obtain a topologically non-trivial coupling. Indeed, without them, the vector $\boldsymbol{h}$ of the Fig. 1 of the manuscript has a constant h_z, such that it cannot encircle the origin by varying the phases, leading to a vanishing Chern number. Nonetheless, the geometric aspects of the adiabatic dynamics detailed in our manuscript (namely Eq. (57, 68)) holds for these simpler models. In those models, the Berry curvature keeps the same sign along the diagonals $\phi_1 = \phi_2$, even though it vanishes on average over the entire torus due to the zero Chern number (see Fig. 1 of this response). As a result, considering $\omega_1 = \omega_2$ (as it is done in the references) and a well chosen initial phase leads to a non-zero average drift. This drift is not topologically quantized, since the average value of the Berry curvature along this line can take any value. Besides, the rate and direction of pumping strongly depend on the initial phase. This is similar to the notion of geometric pumping discussed in the Thouless pumping literature. In the case of topological coupling however, by considering an incommensurate ratio between the frequencies to sample the entire torus, the pumping rate is topologically quantized and robust against the choice of initial phase distribution. We had a paragraph in Sec. 3.2 discussing this aspect.

- *Considering the previous question, it seems that the intriguing physics of the model described in equations (5) arises from a conical intersection, which leads to non-trivial topology. Conical intersections are a well-established topic in molecular and chemical physics, with their associated gauge structures extensively studied. I wonder whether the analysis presented in this paper could be linked to this broader area of physics.*

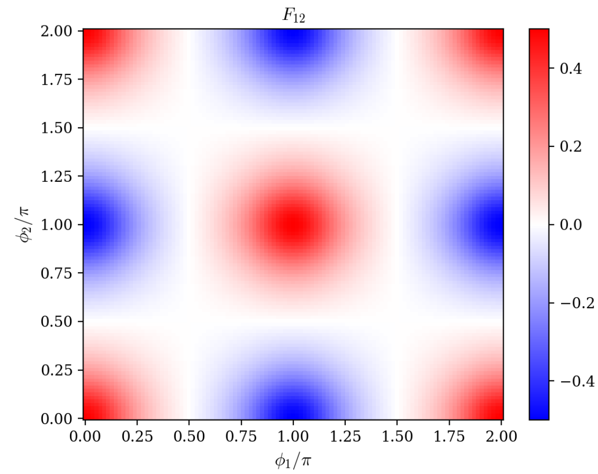

FIG. 1. Berry curvature of the model with $h_z = \Delta/2$.

▷ We thank the referee for the relevent comment. Indeed, the geometrical and topological aspects of the dynamics originates from a conical intersection, since these intersections are sources of Berry curvature like a magnetic monopole is a source of magnetic field. On Fig. 1(b) of the manuscript, the conical intersection corresponds to the origin. However, the presence of a conical intersection is not enough to guarrenty a topological coupling. The Hamiltonian vector $\boldsymbol{h}$ needs to encircle it when the phases are varied (Fig. 1c) to get a non-zero Chern number. In molecular physics language, the dynamics of the nuclei in phase space needs to explore a two-dimensional surface enclosing the conical intersection in order to be sensitive to its topological (and not only geometrical) property. Besides, we emphasize that we chose to focus on the situation where the dynamics is confined far away from those conical intersection. This requirement is the condition of adiabaticity. Very interesting and different physics happens when the dynamics in phase space approaches the conical intersection. This will be studied in a separated work on the case of coupling to the quadrature of the modes. We add a sentence in the end of Sec. 2.1.1 mentioning the relation between topological coupling and conical intersections.