Reply Report 2

**SciPost_202401_00039v1**

Title: Enhancement of stability of metastable states in the presence of Lévy noise

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Dear Anonymous Referee,
thank you very much for your thorough report. We greatly appreciate your valuable comments and suggestions. In particular, we are pleased to acknowledge the positive feedback reflected in your detailed review.

Indeed,
Anonymous Referee 1 states that:
"... An exact expression for the NLRT is derived…. The solution reveals the interesting phenomenon of noise enhanced stability,… The study is carefully performed and mathematically sound…".
Finally, the Referee 2, asking for minor revision, states that: "The manuscript is thus certainly suitable for publication in principle, but I recommend that the authors consider the following comments…".

## Comments

The Referee writes:
1 – Objection
"1. The introduction on the problem is too generic. The problem of escape from metastable states driven by Levy-type noise has attracted a considerable amount of theoretical research over the last two decades. The authors write that "there are a lot of numerical results and some analytical approximations [3,32,35]." But there are lot more than 3 articles that are relevant here (see, e.g., the references discussed in Ref. [35]). This does not have to be exhaustive, but it would be useful to know the different technical approaches used previously and the results found."

Our response:
1 - Reply to objection

1. The Referee is right. We have thoroughly rewritten the introduction to place the manuscript's subject in a broader and more scientifically appropriate context, incorporating additional relevant material. Specifically, we have emphasized the key question under investigation. Additionally, we provide a brief review of theoretical studies on the problem of escape from metastable states driven by Lévy noise, published over the past two decades. This review includes extensive research conducted through both numerical simulations and analytical approximations, with relevant citations (see references [3, 41, 44, 46-55]). Additionally, we have revised and improved the abstract for greater clarity and precision.

The Referee writes:
2 - Objection
2. "Likewise, I wonder if the discussion in Section 2 requires some more references. Has this approach been discussed previously for Gaussian noise?"

Our response:
2 - Reply to objection
The referee is correct. We have added new references and properly contextualized them. As mentioned in the introduction and immediately after equation (1), our study applies to the stability index $\alpha$ of the Lévy distribution within the range $0 < \alpha < 2$. In other words, it does not apply to $\alpha$ =2, which corresponds to Gaussian noise. The phenomenon of noise-enhanced stability under Gaussian noise has been thoroughly investigated using different approaches, as highlighted in the new

references [7-15], particularly through studies on the ordinary Fokker-Planck equation and, in some cases, using functional analysis.

The Referee writes:
3 - Objection
3. “The discussion of the ratio $\langle \tau_{NLRT} \rangle / \tau_d$ below Eqs. (28,29) is confusing, because $\tau_d$ appears without any proper introduction. It would be clearer if first the behaviour of $\langle \tau_{NLRT} \rangle$ would be discussed by itself together with the relevant figures (Fig.2b and Figs. 3a,b). Then, $\tau_d$ could be introduced (this needs more explanations) and the ratio $\langle \tau_{NLRT} \rangle / \tau_d$ discussed.”

Our response:
3 - Reply to objection
The referee is correct. First we changed $\tau_{NLRT}$ to $\tau_{MRT}$. We have now properly introduced the dynamic time $\tau_d$ immediately after presenting the exact quadrature result in Eq. (29), along with the necessary explanations. Additionally, in the footnote on page 7, we have described the non-normalized behavior of the mean residence time in the metastable state, $\tau_{MRT}(x_0)$, as a function of the noise intensity parameter $D_1$ with fixed $L_2$. This shows the same non-monotonic behavior, including a maximum, though with different scaling on the vertical axis of Fig. 2. In Fig. 3, the MRT $\tau_{MRT}(x_0)$ versus $D_1$, with fixed $L_1$, is shown. Additionally, we have added more physical insights on the new Figs. 2 and 3, along with details on the numerical integration of the Langevin equation (1). Please refer to the updated page 8 of the revised manuscript.

The Referee writes:
4 - Objection
4. “The authors should clarify whether "nonlinear relaxation time" and "mean residence time" are the same quantity. If they are identical, I would recommend to use only one of the two terminologies throughout the manuscript.”

Our response:
4 - Reply to objection
The referee is correct. We have consistently used the term “mean residence time” throughout the revised manuscript.

The Referee writes:
5 - Objection
5. “The analytical results should be supplemented by numerical results. It is straightforward to simulate Eq.(2), thus numerical confirmation should be provided.”

Our response:
5 - Reply to objection
We thank the Referee for the suggestion, which we have implemented by complementing the analytical results with numerical simulations of Eq. (1). We found excellent agreement between the exact theoretical results of Eq. (29) and the numerical simulations of Eq. (1), as demonstrated in Figs. 2b and 3b, thereby providing strong numerical confirmation.

The Referee writes:
6 - Objection
6. “The authors mention that the enhancement of stability of metastable states has been observed previously for Gaussian noise. How does $\tau_{NLRT}$ differ quantitatively from this case? It would be important to understand how the stability is affected by the non-Gaussianity of the noise.”

Our response:

6 - Reply to objection

First we changed $\tau_{NLRT}$ to $\tau_{MRT}$. Then, on page 8 we described how $\tau_{MRT}$ with non-Gaussian noise differs qualitatively and quantitatively from the case of Gaussian noise. In particular, we note that in the limit $D_1 \rightarrow 0$, and for unstable initial position of the particle, there is a divergent behavior of $\tau_{MRT}(x0)$ with a Gaussian noise source, see Refs. [7–13, 15]. For Lévy flights, however, $\tau_{MRT}$ (x0) exhibits a finite, nonmonotonic behavior as a function of the noise intensity parameter $D_1$, with finite asymptotic values in the limit $D_1 \rightarrow 0$. Due to the heavy tails of the distribution, a particle spends a finite amount of time in the metastable area even in the limit $D_1 \rightarrow 0$. For very large noise intensity parameter, in the limit $D_1 \rightarrow \infty$, the normalized MRT follows a power-law behavior as a function of the noise intensity parameter, see Refs. [3,4].