Answer to Referee I

We thank the Referee for a very professional and useful Report which definitely helped to improve the presentation of the manuscript. Below we present our response to Referee's comments and the list of changes in the manuscript.

1. *I feel that the presentation could be greatly improved with a relatively small effort, so as to make the paper more accessible to non-specialists. I am referring in particular to the introduction: for instance, the sentence "This balance favors long-distance resonances even at small hopping which may result in the sparse, non-ergodic extended states" is virtually incomprehensible for a non-specialist, since "resonance" and "non-ergodic state" have not yet been defined, even vaguely.*

   We agree with the referee that the use of jargon should be avoided and the notion of 'resonance' and its implication for the structure of wave functions should be explained. We added a paragraph in the Introduction about this issue. Modified text is shown below in red.

   large number of sites at this distance. The two sites can be simultaneously populated in a given wave functions only if they are in resonance with each other (i.e. the difference in their energies is of the order of or smaller than the transmission matrix element). Therefore, the balance between a small transmission matrix element and a large number of sites makes significant the probability of such resonances at large distances between the sites and thus favors sparsely populated but extended wave functions. At a certain range of parameters such wave functions may form a *non-ergodic extended state* [6] in which populated sites form a fractal.

   In addition, we added some more explanations closer to the end of Introduction, see 2nd paragraph in page 4 of the current manuscript.

   The transition from the ergodic to the non-ergodic extended states (the *ergodic transition* at $\gamma = \gamma_{ET}$ [10, 7, 11, 12]) happens as a true phase transition in all the Rosenzweig-Porter models. However, in contrast to the localization transition, it leads to a qualitative change of the *global* density of states (DoS) $\rho(E)$. Namely, in the non-ergodic phases the DoS in the thermodynamic limit tends to a distribution of diagonal entries which is N-independent, while in the ergodic phase $\rho(E)$ is strongly size-dependent.

2. *I found several misprints along the manuscript. For example,*
   - Page 2, "matrix elements which higher moments" -> "matrix elements whose higher moments"

- Page 5, under Eq. (8), "anti-ommuting"
- Page 6, under Eq. (15) "quatric" -> "quartic" (2 occurrences)
- Page 10, under Eq. (33), "is the scaling function" -> "is a scaling function".

We did a spelling check and corrected typos.

3. *Punctuation is missing at the end of most displayed equations.*

We corrected the punctuation.

4. *In Eq. (11), the diagonal Lévy terms are discarded on the basis of their being subleading (for large N) with respect to the Gaussian terms. While this may be legit, I doubt whether it is necessary — they might as well be taken into account with limited modifications, e.g. possibly an extra factor $\sqrt{\boxed{2}}$ with respect to the off-diagonal terms.*
This is not entirely pedantic, since later in the paper the Lévy limit is explicitly considered, and in that case the Gaussian diagonal terms are not there at all.

By setting $\gamma=1,\ W=0$ in the solution for the Levy-RP matrices we neglect the diagonal entries in the pure Levy matrices. However, since there are ~ N diagonal entries and ~ N^2 off-diagonal ones and all entries are identically distributed in the Levy matrices, setting all the diagonal entries equal to zero results in the correction to the spectral densities of Levy matrices which is of the order of $1/N$. Such corrections are beyond the saddle-point approximation we used in this calculations. This is evident from the fact that this approximation does not give the oscillations on the top of the semi-circle for $\mu=2$. For this reason we did not write in the paper corrections of order $1/N$, in particular, the corrections pointed out by the Referee. However, we added a few lines of the corresponding explanations after Eq.(10)

Furthermore, because there are $\sim N$ diagonal entries and $\sim N^2$ off-diagonal ones, by setting zero all diagonal entries in the ensemble of Lévy matrices one obtains only a $1/N$ correction to the pure Lévy DoS. Neglecting such $1/N$ corrections we can safely obtain the DoS of Lévy matrices by setting $\gamma = 1$ and $W = 0$ in the result for the Lévy-RP ensemble.

5. *At the end of page 6, "This non-trivial step was suggested (for different applications) in Refs. [21, 22] and rarely used since then." While I agree that the functional HS transformation is an*

underestimated computational tool, I also think that it would be relevant to cite here some recent examples of its use for closely related problems: see for instance [10.1088/1751-8121/acdcd3] and [arXiv:2408.10530v1]. The first of these two papers addresses the spectrum of sparse random matrices with row constraints, using the very same supersymmetric technique employed here.

We added these two references suggested by the Referee.

6. *On page 7, under Eq. (20), I noted that the super-vector rotation $\hat{T}$ was not defined anywhere in the manuscript. Although it can be found e.g. in Mehta's book, I think it's important to include it here explicitly, to ensure reproducibility.*

Here we disagree with the Referee. All what is needed for calculations (and its reproducibility) is that the rotation supermatrix $\hat{T}$ is unitary. This is explicitly said in the text of the manuscript.

7. *On page 8, I appreciate that Eq. (25) is not an integral equation, but it's excessive to state that it is algebraic. Even a simple trascendental equation such as $x=\sin(x)$ cannot be called algebraic.*

This is a correct comment. We changed the description by saying that Eq. (25) is not an integral but rather an ordinary transcendental equation.

8. *I've been initially puzzled by the fact that the "spectrum of the sum of two random matrices" is a well-studied problem that can be solved within free probability (aka the "Zee formula"), once the resolvent (or Cauchy-Stieltjes transform) of the two individual matrices is known. Now, the resolvent of a diagonal matrix with i.i.d. entries is easily found, while the resolvent associated to a Lévy matrix was in principle studied already in Ref. [17] of the present manuscript. However, to the best of my understanding (this is after all a mathematical paper), Ref. [17] does not contain the resolvent of a Lévy matrix in closed form. I've searched through the recent literature, and concluded that an explicit expression of such resolvent indeed is yet to be found — which makes the free probability prescription inapplicable to the Lévy-RP case. It may be worth highlighting this, since it makes the results of this paper even more remarkable (but of course it's up to the authors to decide).*

We are not sure we fully understood this comment/suggestion. However, we emphasized in the text that our final result for pure Levy matrices is identical to the one of Ref.[20] in the updated manuscript and is not identical with the result of Ref.[18]. We also emphasized the power of our

SUSY method compared to the traditional mathematical tools of Ref.[20] which requires much more efforts to arrive at the same result.

case $\rho(E)$ is N-independent in the large-N limit. It was obtained using the traditional mathematical tools and the proof of the formula equivalent to Eqs.(25),(27) involves rather long chain of arguments. The power of supersymmetric calculus used in the present paper makes it possible to reach the same result much faster.

Notice also that the result of Ref.[18] for $\rho(E)$ for Lévy matrices is not identical to our result (e.g. it requires a solution for two unknown functions) though numerically they are very close.

[18] P. Cizeau and J. P. Bouchaud, *Theory of lévy matrices*, Phys. Rev. E **50**, 1810 (1994), doi:10.1103/PhysRevE.50.1810.

[20] G. Ben Arous and A. Guionnet, *The spectrum of heavily tailed random matrices*, Commun. Math. Phys. **278**, 715 (2008), doi:10.1007/s00220-007-0389-x.

9. *In general, the mere knowledge of the spectrum of a random matrix is not a sufficient proxy of the nature of its eigenstates — one rather needs additional knowledge of two-point functions and/or inverse participation ratios.*
In this manuscript, the newly found spectral density of the Lévy-RP ensemble exhibits a qualitative change in correspondence of $\gamma=1$, from which the authors deduce that "the mean spectral density [...] is sensitive to the transition between the ergodic and the fractal non-ergodic states". However, it seems to me that the whole claim is convincing only in view of previous works on the phase diagram of the model, notably Ref. [11], where $\gamma=1$ was shown to mark the onset of non-ergodic states.
I simply suggest to the authors to recall e.g. in Section 2 these basic facts about the phase diagram of the Lévy-RP ensemble, rather than simply assuming that the reader is already familiar with them.

We added a sub-section 2.2 in which we recall the phase diagram of the Levy-RP model.

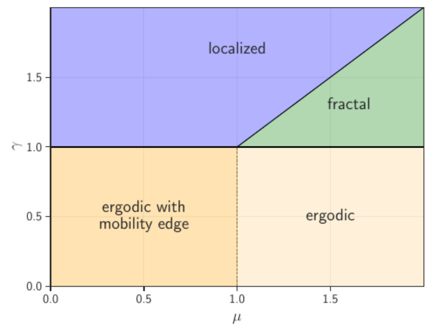

Figure 2: **Phase diagram for the Lévy-RP model.**

2.2 Phase diagram of Lévy- RP matrices

Before coming to calculations of the mean DoS, we would like to recall the main facts about the phase diagram of Lévy-RP matrices mostly following Ref.[12]. The phase diagram in the region of interest $0 < \mu < 2$ is presented in Fig.2. The different phases are identified from the statistics of eigenvectors $\psi_n(i)$ which may be ergodic (the inverse participation ratio $I(N) = \sum_i |\psi_n(i)|^4 \sim N^{-1}$), localized ($I(N) \sim N^0$) or fractal, or extended non-ergodic, ($I(N) \sim N^{-D}$, with $0 < D < 1$). There are two ergodic (E) phases: the one for $1 < \mu < 2$ in which eigenvectors are ergodic at all eigenstate energies E_n and another one for $0 < \mu < 1$ where there is a mobility edge (ME) E_0 beyond which, for $|E_n| > E_0$, the eigenvectors are localized. It is remarkable that all three phases meet at the tricritical point $\gamma = \mu = 1$.

In this paper we are concerned with the simplest spectral statistics which is the mean DoS $\rho(E)$. We show that $\rho(0)$ experience a transition from a N-independent value for $\gamma > 1$ to the rapidly decreasing with increasing the matrix size N value for $\gamma < 1$. According to Fig.2 this is associated with the ergodic/non-ergodic transition in the eigenvector statistics. Notice that $\rho(E)$ is insensitive to the transition from the fractal to the localized phase that happens at $\gamma = \mu$, $1 < \mu < 2$.

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