## **RESPONSE TO REFEREE 1**

Referee 1: The Authors study both static and dynamic properties of self-bound droplets in twodimensional and quasi-two-dimensional geometry. The Gross-Pitaevskii equation for the droplets is solved both numerically and by using the variational technique.

Overall the paper is well-written and the obtained results seem to be credible. The considered problem is relevant to experiments with ultracold atomic gases. The level of the research is appropriate for the Sci Post journal. Nevertheless, I do not find that in the present form, the presentation is clear enough to allow the publication in its present form.

Response: We would like to sincerely thank the referee for their commitment of time and effort to evaluate our manuscript. We are very happy to see that they consider our work relevant, and in the following we will address all the issues raised in detail. At the end of the reply letter a list of changes is appended.

Referee 1: 1. It is said in the Abstract that "One of our main findings is that droplets can become substantially extended when their binding energies become small ...". Phrased this way this result seems trivial, as this is a rather general property of bound states, even not necessary droplets. Actually, one might rather check if the binding energy is inversely proportional to the square of the droplet's size.

Response: We thank the referee for their remark. Indeed, the size of bound states rapidly increases upon decrease of their binding energy which is a general feature of these type of states. This can be readily seen by employing approximate wavefunctions for the bound state, such as the WKB semiclassical approximation. For the droplets, in particular, the relation between their extent and binding energy can be straightforwardly extracted from the variational ansatz described by Eqs. (14), (16) in the main text. This leads to the conclusion that the binding energy scales inversely proportional as  $\sigma_r^{-2}$  along with the logarithmic term  $\sigma_r^{-2} \log(\sigma_r^{-2})$ , where  $\sigma_r$  is the droplet's size. In the case of quasi-2D droplets, there is an additional term, proportional to  $\sigma_r^{-4}$ , stemming from the high density LHY correction in the energy functional [Eq. (9) in the manuscript].

As per the suggestion of the referee, we have modified the statement in the abstract, mentioning that droplets become substantially extended upon transitioning from negative-to-positive averaged mean-field interactions. Following this transition, the binding energies feature a significant reduction which is approximately inversely proportional to the square of the droplet size (see also the list of changes).

Referee 1: 2. The Authors opted for using a micrometer scale for the distances, instead of reporting the results in dimensionless quantities. This makes sense when specific experimental realization is kept in mind. Thus I would prefer to have a detailed explanation of which parameters are used and why.

Response: We thank the referee for their comment. The binary mixture we are investigating pertains to two hyperfine levels of <sup>39</sup>K, namely  $|1\rangle \equiv |F = 1, m_F = -1\rangle$  and  $|2\rangle \equiv |F = 1, m_F = 0\rangle$ . These states have been utilized in the experiments of *PRL*, **120**, 235301 (2018) and Science, **359**, 301 (2017) for realizing quantum droplets in binary bosonic mixtures. Along these lines, the scattering lengths and hence the averaged mean-field interactions that we consider correspond to the values close to the intraspecies Feshbach resonance at 59 G of <sup>39</sup>K, where droplets have been experimentally observed. Moreover, the box length across the z-direction of the quasi-2D geometry is of the order of tenths of microns, which is a typical confinement length considered in *Science*, **359**, 301 (2017). The connection with the experimental realization of droplets is now highlighted in the first paragraph of Sec. 2 in the revised manuscript, see also the list of changes.

Referee 1: 3. I am a little bit puzzled by the geometry used by the Authors. It is said that hard-wall boundary conditions are used. There are two issues with that. In the 2D plane it is more common to use periodic boundary conditions for describing the gas-like unbound state. For the finite-size bound state, the boundary conditions are not important if the size of the bound state is small compared to the box size. This should be mentioned. Instead it is not obvious why the hard-wall condition is used in the transverse direction. Using a harmonic oscillator would sound to be more natural to me. So I would prefer to have a stronger argumentation of that particular choice.

Response: We thank the referee for their insightful remarks. Indeed, the hard walls along the x-y plane are placed sufficiently far away to ensure that they do neither impact the droplet ground states nor their nonequilibrium dynamics induced by the used quench protocol. The choice of the hard walls has been mentioned just before Sec. 2.1, and it is emphasized once again within the revised manuscript in Sec. 4 to avoid any confusion, see also the list of changes.

Additionally, there are two main reasons for choosing hard wall boundary conditions in the transverse (to the plane) direction. First, in such a quasi-2D geometry (i.e. in the absence of a harmonic trap) the LHY correction to the mean-field energy of a binary bosonic mixture is well-established and has been extensively studied theoretically, for instance in *PRA* **98**, 051603(R) (2018) and *PRA* **98**, 051604(R) (2018). Moreover, in the limit where transverse excitations become suppressed compared to the in-plane ones, an analytical solution exists for the LHY term, which has been utilized in our analysis as well. Second, versatile optical box traps are nowadays routinely implemented alongside harmonic potentials in contemporary ultracold atom experiments, see e.g. *Nat. Phys.* **17**, 1334 (2021). This is possible by means of spatial light modulators and digital micromirror devices. Consequently, cylindrical box traps, pertaining to our quasi-2D geometry, can be readily realized experimentally, see for instance the experimental Ref. *Nature* **620**, 521 (2023). To clarify the above arguments, a relevant comment has been inserted in the revised manuscript (see also the list of changes).

Referee 1: 4. Section 2 starts with an overview of 3D experiments. Immediately after an "experimentally verified condition" is given, which involves  $na^2$ , which seems to be more typical to 2D. Please clarify the dimensionality.

Response: We thank the referee for their important comment. The experimentally verified condition that the referee indicates refers to the density ratio of the two components in the droplet regime in 3D. In particular, the density ratio is set by the ratio of the square roots of the 3D intraspecies scattering lengths, i.e.  $n_1/n_2 = \sqrt{a_{22}/a_{11}}$  [*Phys. Rev. Lett.*, **115**, 155302 (2015)]. Upon utilizing the latter relation the binary mixture in the droplet regime is well described by an effective single component equation. A clarification along these lines is provided in the revised manuscript (see also the list of changes).

For our description of the binary mixture in the quasi-2D and 2D geometries, we employ a concrete approach. Specifically, the scattering lengths and the trapping potentials refer to three dimensions. From these we can deduce the relevant equations of motion for the quasi-2D and 2D droplets by means of the pertinent LHY correction [Eq. (8)], and the mapping of scattering lengths between 3D and 2D [Eq. (3)] respectively. In this way, a direct connection between our results and the 3D experimentally relevant parameters is always maintained.

Referee 1: 5. Section 2.1, I think that it would be beneficial for the clarity of the presentation to have two subsections, one on 2D, the other on the quasi-2D.

Response: We thank the referee for their suggestion. Section 2.1 is now split into two subsections referring to the 2D and quasi-2D descriptions respectively (see also the list of changes).

Referee 1: 6. Equation (7) should be introduced better, with detailed explanations of what are the main assumptions which are needed for obtaining this equation.

Response: We thank the referee for their remark. The  $\chi$  parameter in Eq. (7) appears in the derivation of the LHY term in the quasi-2D geometry, as outlined in Ref. *PRA*, **98**, 051603 (2018). At this point no assumption is made, and  $\chi$  defines just the ratio of the characteristic intraspecies interaction energy of the mixture at zero temperature,  $\sim g_{11}n_1 + g_{22}n_2$ , with the energy of the box in the tightly confined dimension,  $\epsilon_0$ . More details regarding the definition of this ratio are included in the revised version of the manuscript (see the list of changes). The only assumption that we make is for the rightmost hand side of Eq. (7), where we consider a fixed density ratio between the two components. This condition is valid in the case of droplets and has been further clarified in the revised manuscript.

Referee 1: 7. I am puzzled by the statement that the relative error during the time evolution is  $10^{-14}$ . Firstly, the way it is written it is not clear if the whole time evolution is intended, or a single time step. Secondly, this number is pretty close to machine epsilon for a double and is smaller than that for a float.

Response: We thank the referee for their comment. The relative energy error refers to the entire time-evolution. To avoid any possible confusion, the relevant sentence and error values have been modified in the revised manuscript (see also the list of changes). Moreover, all numerical computations are carried out with double precision floats. Therefore, even if the relative energy error is close to the respective machine epsilon,  $2.22 \times 10^{-16}$ , our error accumulation is not yet limited by machine precision.

Referee 1: 8. Beginning of Section 3, I would prefer here and in other places to see also the physical terms added to  $\delta a, L, \chi$ , etc.

Response: We thank the referee for their suggestion. We have now added comments on the physical terms of the parameters  $\delta a$  and  $L_z$  at the beginning of Sections 3 and 4, see also the list of changes. The physical terms of other quantities were also introduced wherever appropriate (see the list of changes).

Referee 1: 9. "is small ... namely  $\leq 0.2$ ", it would be more natural to have  $\lesssim 0.2$ 

Response: We agree that this is a more appropriate notation and have now used it in the revised manuscript, see also the list of changes.

Referee 1: 10. Figures 3a and 3b, the blue line is non-monotonous, why is that? Does that mean that at the maximum the flat-top regime disappears or that is a different effect?

Response: We thank the referee for their remark. The non-monotonous behavior of the energy per particle and the droplet width as a function of  $L_z$  in Fig. 3(a)-(b) at  $\delta a = -1.94 a_0$  originates from the interplay between the mean-field (MF) energy,  $|E_{MF}|$ , and the first order LHY correction energy term,  $\left| E_{LHY}^{(1)} \right|$ , see in particular Fig. 2(a) in the main text. Specifically, for  $L_z \gtrsim 0.135 \,\mu m$ , it holds that  $\left| E_{LHY}^{(1)} \right| < |E_{MF}|$ . Since the MF contribution acquires more attractive values as  $L_z$ increases, the droplet becomes more tightly bound, and therefore the energy per particle becomes more negative. On the other hand, for  $L_z \lesssim 0.135 \,\mu m$ , these two energy contributions (mean-field and LHY) are approximately comparable and negative, and thus the energy per particle remains almost constant. This is the origin of the non-monotonous behavior of the droplet's energy per particle and spatial extent at  $\delta a = -1.94 a_0$ . Let us finally note that the flat-top feature is not affected by the above-discussed non-monotonous behavior and it is maintained for any  $L_z$  and  $\delta a = -1.94 a_0$ . A relevant comment has been added in the revised manuscript (see also the list of changes).