Reply to the Referee

This is a well-written manuscript about a new method to detect dark and grey solitons in numerical solutions of the nonlinear Schroedinger equation starting from a simple idea that is based on the powerful inverse scattering transform. Section 2 where the soliton indicator is introduced is very clear and easy to follow. The referencing of previous work appears adequate.

We thank the Referee for acknowledging the novelty and importance of our work in the context of the study of soliton excitations in the nonlinear Schrödinger equation.

I was a bit confused by Sec. 3.1 where empirical thresholds ϵ_- and ϵ_+ given by numerical values are introduced while no reference is made to the theoretically motivated threshold defined in Sec. 2, $\epsilon = \pi^2/(4L^2\sqrt{gn_0})$. As the argument in Sec. 2 appears to make perfect sense, why not use the numerical studies in Sec. 3 to validate it (or understand in which situations it may fail)?

The referee is right, it would be tempting to extrapolate the analytical results of section 2 to more general situations. However the analytical value of ϵ obtained in section 2 relies on the rather strong assumption that the background is uniform (except for the dip of the soliton), which is equivalent to know in advance the precise value of the speed of sound (or the size of the gap in the Lax spectrum). The purpose of section 3 is precisely to extend this method to a case where we do not have beforehand this information. To achieve this we decided to benchmark systematically the predictions of our soliton indicator against the choice of ϵ which led us to define the upper and lower threshold values $\epsilon_\pm$ associated to a confidence interval on the results. We have checked that for a few number of solitons the analytical criterion and the empirical method give the same results.

While the proposed method appears to be potentially useful for numerical studies of generalized hydrodynamics, the introduction also references previous work where solitons had to be identified from experimental data (Ref. [46]). I'd like to encourage the authors to consider and comment on whether their approach could be extended to deal with missing information, i.e. specifically the situation where only density but no phase information is available.

We thank the referee for raising this interesting point. We would like to point out that similar methods have already been used to detect solitons in waves propagating in a water tank, see Ref. [11] of our work. Although the setup is very different from quantum fluids it is described by similar equations, and the Lax spectrum can be studied. The key point is that the measurement of the elevation of the water surface gives directly access to the amplitude of waves and not to the square modulus as in quantum gases. As we mention at the end of our conclusion our methods could be tested on experiments dealing with light propagating in non-linear media, also described by the non-linear Schrödinger equation, for which simultaneous measurement of the intensity and phase of the field is possible (see Refs. [6] and [54]).

To test whether our approach can be extended to deal with missing information, specifically the situation where only the density or the phase is known (but not the two), we have computed the Lax spectrum for a single gray soliton and five solitons cases (described by Eqs. (3) and (8) of the main text). In Fig. A1, we compare the Lax spectra obtained using the full wavefunction ψ, its absolute value $|\psi|$ (no phase) and $\psi/|\psi|$ (only phase), in the Lax operator, for the two cases. Comparing the Lax spectra we see that if part of the information is missing the results are quantitatively changed. Using only the absolute value a single soliton gives birth to two shallower (faster) solitons with opposite eigenvalues (or velocity), while using only the phase a single shallower soliton is present. Comparing the cases with a single and five solitons these general features seem to appear also for states with multiple solitons. This suggest that the measurement of the phase would be sufficient to count the number of solitons and know their direction of propagation, while the measurement of the density would only give the total number of solitons, being half the number of eigenvalues in the gap. However it is not guaranteed that there is a simple way of relate the Lax spectrum computed only from the phase information to the initial one.

Investigating this problem is certainly interesting, but goes far beyond the scope of this work, that aimed at demonstrating the practical usefulness of the Lax spectrum to study solitons in far from equilibrium systems.

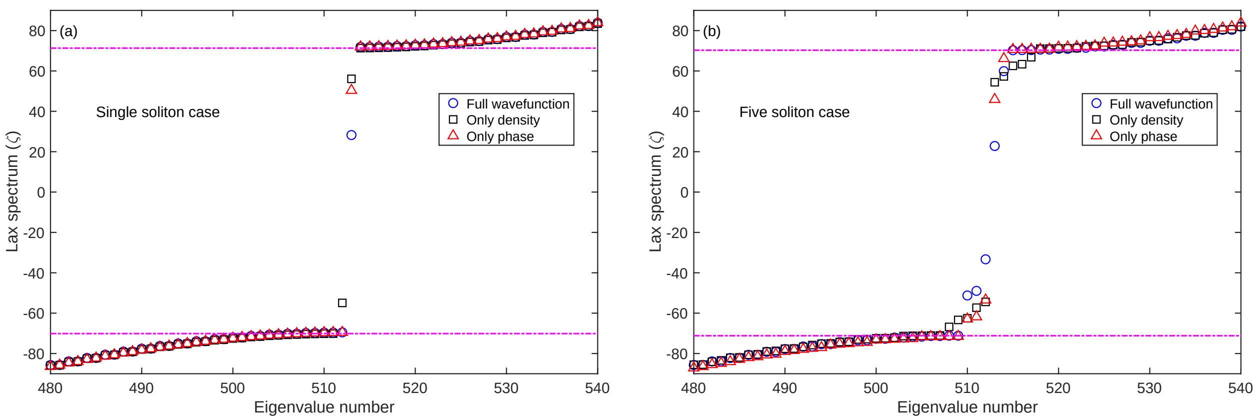

Figure A1: a) Lax spectrum for single gray soliton state with angle $\phi = -\pi/8$ b) Lax spectrum for $N_{\mathrm{imp}} = 5$ solitons. In both plots open blue circles indicate the Lax spectrum calculated using the wavefunction ψ, black square markers correspond to the Lax spectrum of $|\psi|$ and red triangles to $\psi/|\psi|$. The two dashed-dotted magenta lines correspond to the gap boundaries computed for the initial spectrum with $\epsilon_0 = \pi^2/(4L^2\sqrt{gn_0})$.

Other comments:

- *Please add a discussion relating the empirical thresholds ϵ_- and ϵ_+ to the theoretical threshold ϵ of Sec. 2. Moreover, it would be potentially more instructive and generally useful to give the thresholds in unit of ϵ rather than just as bare dimensionless numbers (which only make sense for particular chosen numerical parameters).*

  We thank the referee for this comment. In the updated manuscript, we have added a discussion related to our lower and upper bound of the thresholds, at the beginning of section 3. We now report the values of $\epsilon_\pm$ in unit of $\epsilon_0 = \pi^2/(4L^2\sqrt{gn_0})$, the analytical threshold.

- *The blue solid line in Fig. 6a for eigenvalues identified by threshold ϵ_+ appears to zero for most bins. Does that mean that this threshold fails? Please discuss the implication, or correct the plot if this is just a mistake*

  We thank the referee for raising this important point. The blue solid line describes the extra eigenvalues detected only by the threshold ϵ_+, in addition to the ones detected by threshold ϵ_-. The fact that the curve is zero for most bins shows that these extra eigenvalues are concentrated near the edges of the spectrum, and concern only the fastest solitons. We have rephrased the caption of the figure to clarify this point.

- *The same figure caption mentioned cyan lines corresponding to some numerical factor multiplied to c. What is the significance of the particular factor? Why were these lines included in the plot? Please* *clarify!*

For our numerical simulation of a dilute gas of soliton pairs, we have found that the relative difference between the average number of detected and imprinted solitons $\bar{\Delta}$ is consistent with the assumption that all solitons imprinted with phases up to $|\phi| = 0.97 \times \pi/2$ are correctly detected, as reported in Fig.5 (c). We report these lines in this figure to show that this assumption is supported by the analysis of the histogram: all eigenvalues with phases (velocities) up to $0.97 \times \pi/2$ are correctly detected and only faster (or shallower) soliton are missing. The region between the cyan line and magenta line evidences the eigenvalues corresponding to solitons that are difficult to detect with our method (and probably any other method).