

Response to the referee reports for submission 2409.08208v1

Dear editor,

thank you very much for sending the two referee reports on our paper “Higher-genus Fay-like identities from meromorphic generating functions” after we submitted to SciPost. We are grateful to the two referees for carefully reading our manuscript as well as their comments, suggestions and criticisms. Please find a point-by-point response below, where we hope to provide answers along with indicating the corresponding modifications in the draft.

Referee 1:

Referee comment: I have a hard time accepting definition 1 to be a definition. It only says that there exists a one-form $K(z, x)$ and that this one-form is unique, but it does not say how it looks like. This might be acceptable to mathematicians, but for a publication in a physics journal one expects a proper definition.

Response: In reference [27], Enriquez’ connection has been proven (and thereby defined) as the unique solution to the monodromy and residue constraints in our eqns. (2.2a/b), so we are considering the unique formal mathematical object. The constraints for the connection form $K(z, x)$ correspond to analytic and algebraic constraints for the kernels $\omega(z, x)$ upon taking expansion (2.3) into account. For proving validity of the relations among different kernels $\omega(z, x)$ in the current article, those constraints are sufficient. Constructing an explicit representation for $K(z, x)$ is conveniently done through constructing a representation for the kernels $\omega(z, x)$, for which a uniformization of the underlying Riemann surface must be chosen. In ref. [31], an explicit representation based on the Schottky uniformization is provided.

We do agree with the referee that a **Definition** is not the correct environment to introduce Enriquez’ connection. We switched to **Proposition**, quoting the original proposition from the reference. Furthermore, we added a sentence in the paragraph below the proposition, explaining that for the purpose of the current article it is sufficient to consider the connection $K(z, x)$ as a formal mathematical object, whereas an explicit realization is provided in ref. [31].

Referee comment: The authors might want to comment why the only elements of the algebra appearing in eq. (2.3) are the ones with a number of b ’s on the left and exactly one a on the right.

Response: The particular combination of letters (words) in the kernel expansion (2.3) is the consequence of the algebraic constraints (2.2a/b) in Enriquez’ original definition together with the uniqueness of the connection form. It is possible to show that the appearance of any other words would violate one of the two conditions (2.2a/b). However, it is important to note that the current representation is based on assigning the letters a to the A-cycles and the letters b to the B-cycles, where cycles and differential forms are chosen to be normalized in the standard way, which implies that nontrivial monodromies are carried by letters b only.

As this question turns out to be leading to a similar line of thoughts as the first question of the referee, we just added another sentence after eqn. (2.3) catching those readers noticing the peculiarity, stating its origin but do not explain it further.

Referee comment: The authors might want to comment on the meaning of the denominator $1 \otimes b_i - b_i \otimes 1$ in eq. (3.8). A priori one just has the algebra $\mathfrak{b} \otimes \mathfrak{b}$ and no division.

Response: Thanks a lot for spotting this inconsistency. The fraction should be considered as a formal power series in the (power-series-)completion of the algebra $\mathfrak{t} \otimes \mathfrak{t}$. We have added a footnote explaining the situation and linking eqn. (3.8) to eqn. (A.5) in the appendix, where we perform the explicit expansion.

Referee 2:

Referee comment: Grammatically, there is the use of “between” when I would expect “among” in several places in the text. I would suggest using “among” instead of “between” in several places. For example, this sentence in page 3 read better with “among” instead of “between”: “The properties of the connection form (including its flatness) will then imply relations between integration kernels, which in turn lead to functional relations between polylogarithms and associated special values.”

Response: We appreciate this language comment very much and changed accordingly throughout the article.

Referee comment: Equation (7.17) is a concrete outcome of several computations in this work. It would greatly benefit the reader (and myself as a referee) if they could readily check this numerically, so here is my request: Could you include some code to numerically evaluate depth-2 polylogarithms at genus-2, so the readers can check Equation (7.17)? If this is not possible, I would like to know too.

Response: In answering your request, we added a paragraph and a footnote below eqn. (7.17). Our numerical evaluation of higher-genus polylogarithms relies on the explicit representation of integration kernels in ref. [31] and is unfortunately not possible to wrap into a couple of lines of code: we are facing a programming project of larger size.

In order to calculate a polylogarithm numerically, the Schottky sums for the integration kernels have to be evaluated for each point on the integration path ensuring numerical accuracy, summed up numerically. Far more important and difficult is the branch cut management, which is cumbersome already at the second iterated integral (as no general statements about the branch cuts of depth-one polylogarithms can be made currently). At the moment, we are trying to get a handle on using functional relations for integration kernels to improve convergence speeds substantially in order to get faster results.

As now mentioned in the article, the software package shall be published at some point, but given the complexity of the problem, this might be a slightly more distant future.

Thank you very much,
best wishes,

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Artyom Lisitsyn
Yannis Moeckli