Dear Editor,

Please find attached the revised manuscript titled "*Boundary Time Crystals as AC sensors: enhancements and constraints*" We are grateful to the reviewers for their time and insightful comments. We have carefully considered all the feedback and detailed our responses below. We hope with these changes, the article is now ready for publication.

Yours Sincerely,
Dr. Dominic Gribben - on behalf of the Authors

Reply to Report 1

1. Model Explanation:
For readers unfamiliar with time crystals, it is unclear why this model constitutes a time crystal. The statement "Specifically this is known..." following Equation (1) is vague. I suggest expanding this section with additional references or providing a brief derivation in an appendix to substantiate the claim.

We appreciate the suggestion and have expanded the discussion following Equation (1) in order to clarify the nature and terminology of boundary time crystals. Additional references have also been included to provide a broader theoretical context. See paragraph "Specifically this is known as... supporting a time crystal phase.".

2. Discussion of Quantum Fisher Information (QFI):
Extending the discussion of QFI in Equation (4) to include the expected scales and their implications for the sensor's performance would be welcome. For instance, explaining the significance of $F/N > 1$ as a quantum enhancement and why $F/N \approx N$ indicates the Heisenberg limit. Although some of this discussion is already present in the introduction, I believe that moving and expanding it would benefit the coherence of the manuscript. I also think that the introduction of the operator $\hat{L}_g$ seemed unnecessary since it was not used in practice to obtain the results.

To improve coherence, we have expanded the discussion in Section 3 regarding the expected scaling of the QFI and its implications. See paragraphs "The QFI bounds the uncertainty in estimating the value of g... decode the information about g" and "A few bounds of the QFI ... Heisenberg limit with $F = N^2t^2$." Additionally, we agree that the operator $\hat{L}_g$ was not used directly in obtaining our results, and therefore we have removed it.

3. Normalization in Figures:
I would recommend using consistent normalization of the figures. For example, in Figure 1 F was normalized to N^2, while Figures 2 and 5 normalize F to N. Alternatively the authors could justify, in each figure, their choice for the given normalization.

We have ensured consistency in figure normalizations. Now, all figures with QFI data use a uniform normalization of F/N for clarity and comparability.

4. Choice of External Field Orientation:
The external field is assumed to be oriented in the z-direction, as defined in Equation (3). Clarify whether this choice is standard or made for convenience. If it is standard, cite appropriate literature. If not, discuss the implications of considering arbitrary driving directions.

Our observations suggest that choosing the z-direction for the external field results in higher QFI values as compared to the x or y directions. While we lack an analytical proof that this direction is always optimal, we can provide an intuitive justification. The intuition comes from the fact that the time-crystal phase is unstable - most susceptible - to perturbations along the z-direction (according to analytical results within a mean-field treatement [PhysRevB.103.184308 (2021)] or also observed in generalized versions of the model [PhysRevA.103.013306 (2021), PhysRevB.104.014307 (2021)) therefore imprinting a larger response to the applied AC field, as quantified by the QFI. We have now included this discussion in the manuscript, along with references to related studies. See paragraph "Our observations indicate that...a greater precision in its estimation."

5. Entropy Suppression:
Does the parameter ω_0 (coherent driving strength) influence entropy production? Could modifying the driving Hamiltonian, e.g., introducing Heisenberg interactions, suppress entropy production and improve performance?

In our model, the coherent driving strength ω_0 does influence the system dynamics. Increasing it leads to larger entropy production, which however occurs alongside an increase in correlations, as quantified by the variance of the spin magnetization. In this way, a key observation is that both entropy and correlation variance exhibit the same monotonic dependence on ω_0. This implies that increasing correlations (beneficial for the sensor) simultaneously results in higher entropic cost (detrimental to sensing). The interplay between these competing effects does not allow for significant improvements in sensor performance through adjustments in ω_0 alone.

Introducing Heisenberg interactions could be a promising avenue for future research, which we leave as an interesting perspective since the goal of the present work was to highlight the potential of BTCs as sensors, and discuss its main peculiar behaviors. Nevertheless, an interesting direction could be understanding how new interactions in the dynamics - either in the form of coherent dynamics (as e.g. the suggested Heisenberg ones) or in the dissipative jumps - could enlarge the correlations while still maintaining a low - or stable - entropic cost. If possible this has the potential to improve the sensor performance. Therefore, investigating whether modifications of the model can effectively balance these trade-offs remains an exciting direction for further study.

6. Code Availability:
To ensure reproducibility of the results I would suggest, not require, the authors to publish also simulation codes on a public repository such as GitHub. This aligns with the transparency encouraged by the SciPost community and will benefit future researchers working on related topics.

We sincerely appreciate the referee's suggestion regarding the publication of our simulation codes in a public repository such as GitHub. We acknowledge the importance of transparency and reproducibility in scientific research. While we have not shared the codes publicly on GitHub, we are committed to making them available upon request. To reflect this, we have included the following statement in the manuscript:

"The data and codes of this manuscript are available from the corresponding author upon reasonable request."

7. Symbol Usage:
The same symbol S is used for total spin length and entropy. This could confuse readers. Consider using a distinct symbol for the spin operator.

We have removed the usage of S as total spin length, adding explictly its value when needed.

8. Typo:
On page 6. "... they were were generated ... " has too many "were" -s.

The typo on page 6 has been corrected. Thanks for pointing this out.