The authors of the paper

Entitled:

<u>New Exotic Many-body Interference in 2D-Topological Superfluid Fermi Gases: A Non-Adiabatic SU(4) Symmetry Approach</u>

Reply to

The Reviewer Comments

1- I do not see at all the role of so-called SU(4) symmetry. Of course, the system is described using a 4-component spinor (as in BdG formalism) but they do not transform in an SU(4) irrep. What is the role, if any, of SU(4) here ? If it is irrelevant, it should be removed from the paper.

Response

We appreciate the reviewer's concern regarding the $SU(4)$ symmetry which is relevant. This symmetry is manifested through its $4^2-1=15$ generators, which are traceless 4x4 Hermitian matrices. While these generators are not uniquely defined, we have constructed a specific set using tensor products of Pauli matrices $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and the 2x2 identity matrix $I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Specifically, we obtain the following generators:

$\xi_1=\sigma_1\otimes I_2, \xi_2=\sigma_2\otimes I_2, \xi_3=\sigma_3\otimes I_2$, $\xi_4=I_2\otimes\sigma_1$, $\xi_5=I_2\otimes\sigma_2$, $\xi_6=I_2\otimes\sigma_3$

$\xi_7=\sigma_1\otimes\sigma_1$, $\xi_8=\sigma_1\otimes\sigma_2$, $\xi_9=\sigma_1\otimes\sigma_3$, $\xi_{10}=\sigma_2\otimes\sigma_2$, $\xi_{11}=\sigma_2\otimes\sigma_1$, $\xi_{12}=\sigma_2\otimes\sigma_3$

$\xi_{13}=\sigma_3\otimes\sigma_3$, $\xi_{14}=\sigma_3\otimes\sigma_1$, $\xi_{15}=\sigma_3\otimes\sigma_2$.

When the Hamiltonian in equation (5) $:H=\begin{pmatrix}\varepsilon_1&\lambda_k&0&\Delta\\\lambda_k^*&\varepsilon_2&\Delta&0\\0&\Delta^*&-\varepsilon_1&\lambda_k^*\\\Delta^*&0&\lambda_k&-\varepsilon_2\end{pmatrix}$ with $\varepsilon_1=\epsilon_{k\uparrow}-\mu$ and $\varepsilon_2=\epsilon_{k\downarrow}-\mu$ is expressed in terms of these generators as $H=\frac{\varepsilon_1+\varepsilon_2}{2}\hat{\xi}_6+\frac{\varepsilon_1-\varepsilon_2}{2}\hat{\xi}_{13}+\mathrm{Re}(\Delta)\hat{\xi}_7-\mathrm{Im}(\Delta)\hat{\xi}_8+\mathrm{Re}(\lambda_k)\hat{\xi}_1+\mathrm{Im}(\lambda_k)\hat{\xi}_2$, we find it to be a linear combination of SU(4) generators, and it cannot be reduced to SU(2) or SU(3). This demonstrates the presence of SU (4) symmetry in our system. The Dynamic Matrix Approach this symmetry is embedded in calculation of the transition matrices through the evaluation of the time ordering factors.

2- Why do the authors use the terminology "multiphoton" in several places?

Response

The terminology 'multiphoton' is employed to establish an analogy with quantum optics, where it precisely describes processes involving multiple photons. In the context of our Landau-Zener-Stückelberg-Majorana (LZSM) experiments, particularly those utilizing superconducting qubits or other artificial atoms, the driving field, often in the microwave or radio-frequency range, can be conceptually linked to photons, despite operating outside the optical regime. Furthermore, the multilevel LZSM transitions exhibit a conceptual similarity to the behavior of light (photons) traversing a series of transparent mirrors, effectively functioning as beam splitters with associated transmission and reflection coefficients. The subsequent interference effects observed after the 'beams' recombine further support this analogy.

3- Stückelberg oscillations can also be found using Bloch states, see e.g. Phys. Rev. A 92, 063627 (2015)

Response

We sincerely thank the reviewer for pointing out this relevant connection to Bloch state descriptions. We will incorporate the suggested reference, Phys. Rev. A 92, 063627 (2015), as well as other pertinent works that explore Stückelberg oscillations within the framework of Bloch states.

4- I do not understand the sentence "(data) suggest that the system maintains coherence". By definition, there is no decoherence so the system has to remain coherent?

Response

We acknowledge that the term 'coherence' was not the most appropriate choice in this context and apologize for the ambiguity. Our intention was to emphasize that the observation of the reported interference patterns is contingent upon specific parameter ranges. For instance, the clear interferometric fringes depicted in our figures are only obtained within a particular set of conditions. These fringes correspond to regions of maximum transition probability, indicating an optimal parameter regime for observing the predicted interference effects in this regards the range of parameter inducing this can be said in some sense to be in "in coherent regime".

5- Since there is an analytic solution for linear or periodic driving, it would be useful to check the numerical study in some limit.

Response

We sincerely appreciate the reviewer's insightful suggestion regarding the comparison between our numerical simulations and analytical solutions. While we initially omitted this comparison due to space constraints and a perceived lack of necessity, we acknowledge its importance in validating our numerical results.

We have indeed derived analytical solutions for specific limiting cases: after a half period for periodic driving (due to the system's repetitive behavior) and in the long-time limit for linear driving. The numerical simulations, on the other hand, provide a generalized solution applicable at all times.

As requested, we have performed a detailed comparison between the numerical and analytical results in these limiting cases. Figures 1 and 2 demonstrate excellent agreement between the numerical (solid lines) and analytical (dashed lines) transition probabilities in the low and high frequency regimes, respectively.

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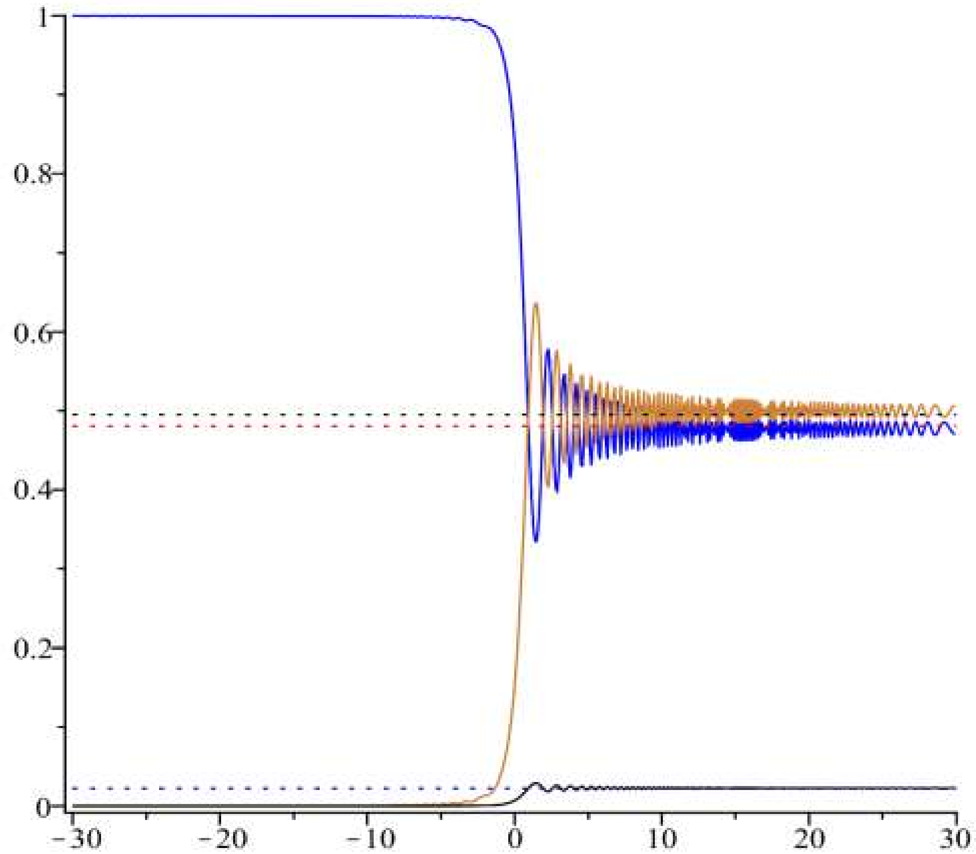

Figure 1: Transition probabilities in the low-frequency limit, comparing analytical (dashed lines, equations 20-24) and numerical (solid lines) results. This verifies the numerical method's accuracy in this regime.

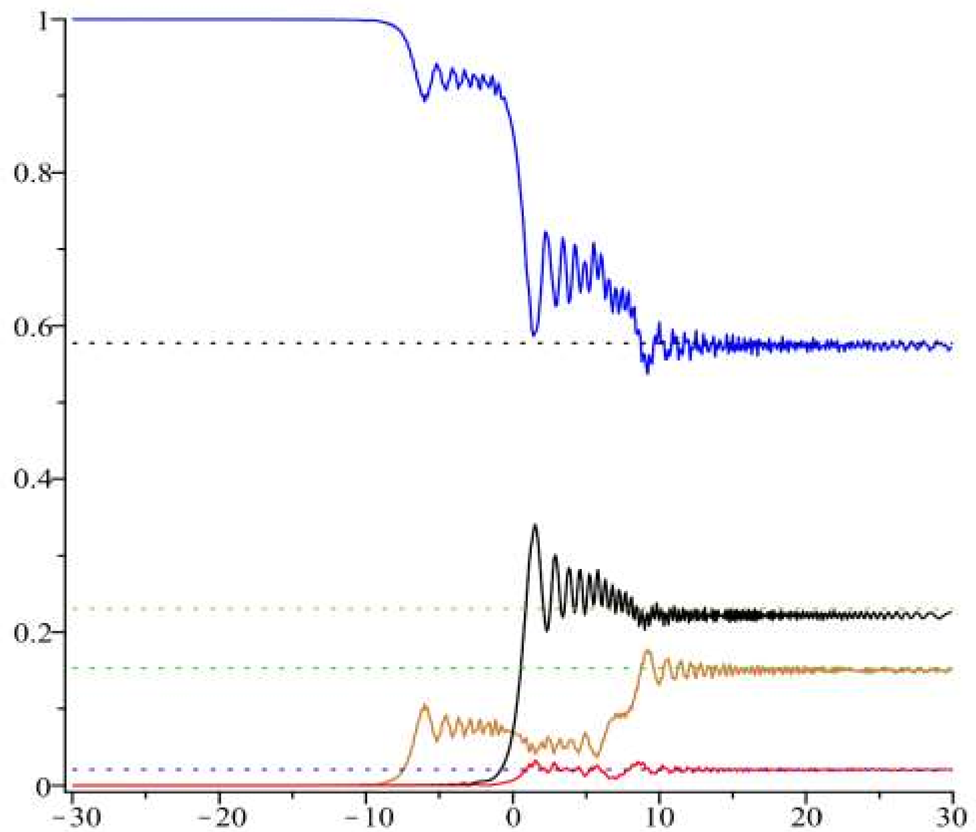

Figure 2: Transition probabilities in the high-frequency limit, comparing analytical (dashed lines, equations 38-42) and numerical (solid lines) results. This verifies the numerical method's accuracy in this regime.

6- The experimental platforms section should be expanded: what would be the condensed-matter system? For a cold atomic setup (where parameters are given), there is no simple equivalent of ARPES, so how could this be checked?

Response

Thank you for your insightful comment. Below, we expand on the possible condensed-matter realization and address the feasibility of an ARPES-equivalent measurement in a cold atomic setup.

Condensed-Matter Platform

A promising condensed-matter system that could exhibit similar Stückelberg interference effects and spin-orbit coupling (SOC)-driven topological transitions is **proximity-coupled topological insulators (TIs) and superconductors (SCs)**. For example:

- **HgTe quantum wells** and **InAs/GaSb heterostructures**, where strong SOC gives rise to topological phases.
- **Iron-based superconductors** (FeSe, FeTe) and **van der Waals heterostructures**, which naturally exhibit topological superconductivity when coupled with a strong SOC substrate.
- **Twisted bilayer graphene (TBG) in the presence of SOC**, where band topology can be manipulated via external gating.

In these platforms, Stückelberg oscillations and quantum interference could be studied through transport measurements (e.g., differential conductance spectroscopy via scanning tunneling microscopy (STM) or angle-dependent magnetoresistance).

ARXPS as an ARPES Alternative in Cold Atoms

In a cold-atom setting, direct ARPES measurements are not feasible due to the absence of a solid-state band structure. However, an **angle-resolved expansion technique (ARXPS, Angle-Resolved Expansion Spectroscopy)** could be employed to extract momentum-space information. This could be implemented as follows:

- **Time-of-flight (TOF) imaging** after releasing the atoms from the trap, capturing their momentum distribution.
- **Band mapping via Raman spectroscopy**, where Raman transitions can selectively probe different quasimomentum states.
- **Bragg spectroscopy**, which allows momentum-resolved measurements by scattering light at a specific angle.
- **Spin-resolved imaging techniques**, enabling the reconstruction of the underlying band structure.

Conclusion

While ARPES is not directly applicable in ultracold atoms, these alternative techniques provide a way to extract momentum-space information and verify Stückelberg interference effects. Expanding on these methods in the revised manuscript will help clarify how our proposed protocol can be experimentally validated in different physical systems.

Experimental Protocol

Interferometry has long been a fundamental technique in metrology, serving various purposes. In this context, we propose a protocol for engineering system parameters, with a focus on spin-orbit coupling (SOC). SOC plays a crucial role in phenomena such as the Spin-Hall effect and topological insulators and contributes to the electronic properties of materials like GaAs, with implications for spintronic devices. Quantum many-body systems involving ultracold atoms provide an ideal experimental platform for precise parameter control, making them well-suited for the study of SOC. Recently, Landau-Zener-Majorana-Stückelberg (LZMS) interference has emerged as a potential approach to achieving perfect population transfer and implementing quantum gates in quantum control and quantum computing.

Our study considers an ultracold **40K^{40}K** atomic gas with a density of **n=5×1012cm−2n = 5 \times 10^{12} \text{cm}^{-2}**, corresponding to a Fermi energy **EF=7πℏkHzE_F = 7\pi \hbar kHz**, determined by the Fermi wave vector **KF=2πnkHz**. The system is polarized by two laser beams with tunable intensities in the transverse (729 nm) and longitudinal (866 nm) directions, initialized with the parameters **Ω=100π\Omega = 100\pi MHz, d0=50πd_0 = 50\pi MHz, ω=0.2Ω\omega = 0.2\Omega, and ϕ=0.5π\phi = 0.5\pi**. By tuning the **s-wave scattering length** via the Feshbach resonance, we can vary **Δ~\tilde{\Delta}** and obtain the results shown in Figures 4 and 8.

Two possible experimental approaches emerge from our study:

1. **Interference Fringe Analysis**: Measuring the distance between the minima or maxima in the Stückelberg oscillations or analyzing the slope of the interference fringes can provide key insights into the system's SOC and other parameters.
2. **Multiphoton Resonance Spectroscopy**: By analyzing the positions of multiphoton resonances, we can extract detailed information about the system's underlying parameters.

Condensed-Matter Realization

A possible condensed-matter platform for studying similar Stückelberg oscillations and topological transitions includes **proximity-coupled topological insulators (TIs) and superconductors (SCs)**. Examples include:

- **HgTe quantum wells** and **InAs/GaSb heterostructures**, where strong SOC drives topological phases.
- **Iron-based superconductors (FeSe, FeTe)** and **van der Waals heterostructures**, where induced superconductivity in SOC-rich materials can give rise to Majorana modes.
- **Twisted bilayer graphene (TBG) with SOC**, where the band topology can be tuned via gating and external fields.

In these platforms, Stückelberg oscillations and quantum interference effects could be probed through **transport measurements**, such as **angle-dependent magnetoresistance** or **differential conductance spectroscopy (via STM or Josephson junction measurements)**.

Momentum-Space Measurements in Cold Atoms

Unlike condensed-matter systems, cold atomic gases do not have a direct equivalent to **ARPES** for momentum-space analysis. However, several alternative techniques can be employed:

- **Angle-Resolved Expansion Spectroscopy (ARXPS)**: This technique involves releasing the atomic cloud and using time-of-flight (TOF) imaging to reconstruct the momentum distribution.
- **Raman Band Mapping**: Raman transitions can selectively probe different quasimomentum states, allowing reconstruction of the effective band structure.
- **Bragg Spectroscopy**: Momentum-resolved Bragg scattering can be used to extract dispersion relations and interaction effects.
- **Spin-Resolved Imaging**: By using state-selective imaging, spin-dependent momentum distributions can be reconstructed.

These methods enable precise validation of Stückelberg interference effects in ultracold atoms and provide a pathway to studying quantum interference in engineered synthetic matter.

Conclusion

Our proposed protocol offers a versatile and effective method for studying and manipulating SOC in ultracold Fermi gases, with potential applications in quantum technologies and metrology. While ARPES-like measurements are challenging in cold-atom experiments, **ARXPS, Bragg spectroscopy, and Raman band mapping** provide feasible alternatives. Furthermore, analogous effects in condensed-matter systems could be explored through transport and spectroscopic techniques.

7- Minor points: there are some typos " turned " -> "tuned ", " Stratonovic " etc.

Response

We sincerely thank the reviewer for pointing out these typographical errors. We will meticulously proofread the entire manuscript and correct all instances of typos and grammatical errors, including 'turned' to 'tuned' and the correct spelling of 'Stratonovich.' Ensuring the accuracy and clarity of our writing is of utmost importance.

8- Minor point: there are issues with some references, e.g. wrong doi for [40]

Response

We sincerely thank the reviewer for identifying the errors in our references. We will thoroughly review and correct all reference entries, including the incorrect DOI for reference [40].

Additionally, we have identified typographical errors within equations (25), (40), and (41) that require correction. We will ensure these errors are rectified in the revised manuscript.