Response to Report A on “Four no-go theorems on the existence of spin and orbital angular momentum of massless bosons”

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Dear Referee,

We thank you for your careful reading and excellent comments. Below we address each comment and describe the accompanying changes in the manuscript.

Referee comment:

> 1) I disagree with a statement that is written in the abstract
> "SAM-OAM splitting is unambiguous for massive particles"
> and also in the introduction
> "For massive particles, the angular momentum naturally and uniquely splits into SAM and OAM."
> and whose justification is contained right after Eq. (17).
>
> I do not think that the split is valid even for massive particles. Here is why.
>
> For a particle of mass M, it is clear that the little group of the standard vector with four-momentum $(M, 0, 0, 0)$ is SO(3). Note that $(M, 0, 0, 0)$ represents a particle at rest with 3-momentum equal to $(0, 0, 0)$. For such particle at rest, the spin-1 matrices are indeed the angular momentum operators. However, this does not mean that the spin-1 matrices are good angular momentum operators for the general case of massive particles out of their rest frame. Rather, one can see in the definition of $\boldsymbol{J}$
>
> $$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{S}. \tag{1}$$
>
> that $\boldsymbol{J} = \boldsymbol{S}$ when $\boldsymbol{P} = \boldsymbol{0}$, as a particular case.
>
> The point is that the Poincare group contains only one kind of spatial rotations, and such transformations have $\boldsymbol{J}$ as their generators.
>
> I refer to the authors to Sec. 10.4.2 of Wu-Ki Tung's book: Group theory in physics, and also to to Sec. 16 of the fourth volume of the Landau and Lifshitz course of theoretical physics: Quantum Electrodynamics, where it says that: "In the relativistic theory the orbital angular momentum L and the spin S of a moving particle are not separately conserved. Only the total angular momentum is. The component of the spin in any fixed direction (taken as the z-axis) is

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therefore not conserved and cannot be used to enumerate the polarization (spin) states of the moving particle."

Reply and Revisions:

In our goal to focus on the massless case, our original manuscript was overly brief in discussing the massive case which, as you point out, has some subtleties to it as well. In summary, there is potential confusion because there are actually two different spin operators in use for massive particles [1], but only one of these commutes with the Hamiltonian. One is the Wigner SAM operator defined on particles with positive energy; this commutes with the Hamiltonian and thus its eigenvalues constitute "good" quantum labels. The other is the Dirac spin which is defined on both the positive (particle) and negative (anti-particle) energy states (i.e. in the bispinor representation). The Dirac spin mixes the positive and negative energy solutions and thus does not commute the Hamiltonian. We discuss this in detail below.

We first note that Foldy [2] gives a simple construction of relativistic massive spin s representations on wave functions $\chi^a(\boldsymbol{r}, t)$ with $2s+1$ complex components. In this representation (in position space) [2] $\boldsymbol{J} = -i\boldsymbol{r} \times \boldsymbol{\nabla} + \boldsymbol{s}$ where $\boldsymbol{s}$ are the spin s matrices which act only on the internal components of the wave function (they mix the $\chi^a(\boldsymbol{r}, t)$ at fixed $(\boldsymbol{r}, t)$). The Hamiltonian is given (neglecting constants) by Foldy's Eq. (23):

$$H\chi^a(\boldsymbol{r}, t) = \int \sqrt{m^2 + k^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r'})} \chi^a(\boldsymbol{r'}, t) d\boldsymbol{r'} d\boldsymbol{k}. \tag{2}$$

Since the Hamiltonian does not mix components of χ and since $\boldsymbol{s}$ does not change $(\boldsymbol{r}, t)$, we have that $[H, \boldsymbol{s}] = 0$, so $\boldsymbol{s}$ is conserved (and therefore so is $\boldsymbol{l} \doteq -i\boldsymbol{r} \times \boldsymbol{\nabla}$). Thus, contrary to the quoted claim by Landau and Lifshitz, it is possible to define SAM and OAM operators for massive particles which are independently conserved (although their claim makes sense in reference to the bispinor/Dirac representation we will later discuss). We note that Foldy explicitly constructs a massless representation for which this holds. Wigner [3] uses substantially more involved methods to show that all representations with the same mass are isomorphic, so that it follows that every massive representation can be put into the form described by Foldy if coordinates are correctly chosen (although it is not *a priori* obvious how to do so). Indeed, one can use Wigner's little group method to explicitly write down SAM and OAM operators which commute with the Hamiltonian and to choose coordinates which put $\boldsymbol{s}$ and $\boldsymbol{l}$ into the simple Foldy form. This is described in Ref. [1] and this SAM operator is called the Wigner spin. We have derived an equivalent form of this for an arbitrary massive vector bundle representation. We describe this construction here and have added it to Section 3 of the manuscript.

Let $\pi : E_m \to \mathcal{M}_m$ denote a vector bundle consisting of all states (k, v) of a particle with definite 4-momentum $k = (k^0, \boldsymbol{k})$ on the mass m hyperboloid where v describes the internal polarization states (which has $2s+1$ DOFs). Let Σ be the Poincaré action on E_m. We seek a decomposition

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{3}$$

where $\boldsymbol{S}$ and $\boldsymbol{L}$ both generate SO(3) symmetries of E_m, and $\boldsymbol{S}$ does not change k. As you point out, $\boldsymbol{J}$ generates the canonical copy of SO(3) in the Poincaré group $\mathrm{ISO}^+(3,1)$ describing the pure 3D rotations (no boost or translation). $\boldsymbol{S}$ and $\boldsymbol{L}$ will certainly not generate that canonical copy of SO(3) since we assume that $\boldsymbol{S}$ and $\boldsymbol{L}$ are both unequal to $\boldsymbol{J}$. Nonetheless, we can still seek other SO(3) actions on E_m. In fact, the Poincaré group contains an infinite number of copies of SO(3) apart from the aforementioned canonical copy, and these show up in the little group method. The little group H_k at k is defined by Eq. (17) in the manuscript as the set of Lorentz transformations leaving k invariant. The canonical copy of SO(3) corresponds to H_{k_0} where $k_0 = (m, 0, 0, 0)$. For $k \neq k_0$, let Λ_k be the unique boost taking k_0 to k, and then we see that

$$H_k = \Lambda_k \circ H_{k_0} \circ \Lambda_k^{-1}. \tag{4}$$

Since the subgroups H_{k_0} and H_k are conjugate (in the group theory sense), they are isomorphic. For each k we obtain a copy of SO(3) in $\mathrm{ISO}^+(3,1)$ which describes an internal SO(3) symmetry of the particles with momentum k. For $k \neq k_0$, many of the elements of H_k are not pure rotations. Indeed, typical elements of H_k will involve boosts to some new k' (with $|\boldsymbol{k}'| = |\boldsymbol{k}|$) followed by a rotation which brings k' back to k. We note that $H_k \neq H_{k'}$ for $k \neq k'$, so no single H_k generates an internal SO(3) symmetry on all of E_m. However, Eq. (4) indicates how such a global action Σ^S can be constructed. For a rotation $R \in \mathrm{SO}(3)$, define

$$\Sigma^S(R)(k, v) = \Sigma(\Lambda_k)\Sigma(R)\Sigma(\Lambda_{-k})(k, v). \tag{5}$$

This action just applies the rotation R in the rest frame of the particle. This action does not change the momentum k, so it describes a global internal symmetry. From this property it is clear that this is not the same SO(3) action generated by $\boldsymbol{J}$, since the latter rotates the 3-momentum. The generator is

$$\boldsymbol{S} = \Sigma(\Lambda_k) \circ \boldsymbol{J}_0 \circ \Sigma(\Lambda_{-k}) \tag{6}$$

where $\boldsymbol{J}_0$ is the restriction of $\boldsymbol{J}$ to the fiber $E_m(k_0)$ of states with momentum $k_0 = (m, 0, 0, 0)$. $\boldsymbol{S}$ is a spin angular momentum operator for a massive relativistic particle (the Wigner spin); we emphasize that it is not equal to $\boldsymbol{J}$. We can choose coordinates $(v_1, \ldots, v_{2s+1})$ for $E_m(k_0)$ (corresponding to the states of a particle at rest) such that $\boldsymbol{J}_0$ acts by the spin s matrices $\boldsymbol{S}_s$, so that

$$\boldsymbol{S} = \Sigma(\Lambda_{-k}) \circ \boldsymbol{S}_s \circ \Sigma(\Lambda_k). \tag{7}$$

In fact, as in the little group method (Ref. [4], Eq. 2.5.5), we can choose coordinates for the other fibers as

$$(k, v_i) \doteq \Sigma(\Lambda_k)(k_0, v_i). \tag{8}$$

Applying Eq. (7) in these coordinates, we obtain the coordinate representation

$$\boldsymbol{S} = \boldsymbol{S}_s. \tag{9}$$

In other words, using the fact there is a unique boost from k to k_0, we can label any internal state according to its corresponding rest frame label. The $2s+1$ component wave functions used in Foldy's representation are sections of the trivial bundle $\mathbb{R}^3 \times \mathbb{C}^{2s+1}$. What we just did was construct a canonical isomorphism between E_m and that trivial bundle, such that under this isomorphism the SAM operator has the simple Foldy form. Said differently, Eq. (5) gives the general definition of the spin angular momentum operator and Eq. (9) shows how it can be described by the spin s matrices if rest frame coordinates are used to label internal states. The orbital angular momentum is defined by

$$\boldsymbol{L} = \boldsymbol{J} - \boldsymbol{S}. \tag{10}$$

It generates an SO(3) symmetry since $\boldsymbol{J}$ and $\boldsymbol{S}$ do.

We note that $\boldsymbol{S}$ and $\boldsymbol{L}$ both generate symmetries which do not change the magnitude of the three-momentum $|\boldsymbol{k}|$. Therefore, they do not change the energy $k^0 = \sqrt{m^2 + |\boldsymbol{k}|^2}$. They thus commute with the Hamiltonian of the free theory so that $\boldsymbol{S}$ and $\boldsymbol{L}$ are both conserved (although this is not necessarily true once interactions are included, which is also the case for nonrelativistic SAM and OAM). Because $\boldsymbol{S}$ and $\boldsymbol{L}$ are both generators of SO(3) symmetries and thus angular momentum operators, one can apply the addition of angular momentum equation and the Clebsch-Gordon sum rule to immediately determine the total angular momentum eigenstates of the particle and their multiplicities, and one sees that these multiplicities agree with those found in nonrelativistic quantum mechanics.

We now turn to a discussion of the Dirac spin operator $\boldsymbol{\Sigma}$ for massive spin 1/2 particles, which is also discussed by Terno [1] and how it differs from the Wigner spin operator just described. Here, we will just show that unlike the Wigner spin, the Dirac spin does not commute with the free theory Hamiltonian. This operator arises in the bispinor/Dirac representation in which one works with an internal state space of dimension $2(2s+1) = 4$ rather than $2s+1 = 2$. The doubling of the internal degrees of freedom corresponds to the inclusion of negative energy solutions which correspond in turn to anti-particles. In this representation, the definite momentum states are elements of the trivial bundle $\mathbb{R}^3 \times \mathbb{C}^4$, where the $\mathbb{R}^3$ corresponds to the momentum $\boldsymbol{k}$ and $\mathbb{C}^4$ to the internal states. At each $\boldsymbol{k}$ there are two states with positive energy E ([5], Eqs. (3.99) and (3.100))

$$|\boldsymbol{k},+1\rangle = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 \\ 0 \\ \frac{k_z}{E+m} \\ \frac{k_x+ik_y}{E+m} \end{pmatrix}, \quad |\boldsymbol{k},+2\rangle = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 0 \\ 1 \\ \frac{k_x-ik_y}{E+m} \\ \frac{-k_z}{E+m} \end{pmatrix} \tag{11}$$

and two states with negative energy $-E$

$$|\boldsymbol{k},-1\rangle = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \frac{k_z}{E+m} \\ \frac{k_x+ik_y}{E+m} \\ 1 \\ 0 \end{pmatrix}, \quad |\boldsymbol{k},-2\rangle = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \frac{k_x-ik_y}{E+m} \\ \frac{-k_z}{E+m} \\ 0 \\ 1 \end{pmatrix}. \tag{12}$$

The Dirac spin operators are ([6], Eq. 3.27)

$$\Sigma^j = \frac{1}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \tag{13}$$

where the σ^j are the Pauli matrices. The Σ^j generally mix the positive energy states with the negative energy states, for example,

$$\Sigma^x|\hat{\boldsymbol{k}}_z,+1\rangle = \frac{(E+m)^2+1}{(E+m)^2-1}|\hat{\boldsymbol{k}}_z,+2\rangle + \frac{2(E+m)}{(E+m)^2-1}|\hat{\boldsymbol{k}}_z,-2\rangle. \tag{14}$$

Thus, the Hamiltonian does not generally commute with $\boldsymbol{\Sigma}$. Since $\boldsymbol{\Sigma}$ mixes particles and anti-particle states, it describes a symmetry of the bispinor space but not of the particle or anti-particle states independently. On these grounds, one may argue that that the Wigner spin operator is the correct SAM operator for massive particles; Terno [1] gives additional arguments supporting this conclusion. Nonetheless, the Dirac spin operator is an important and commonly used theoretical tool, and we have altered our paper to avoid confusion. We note that when Landau and Lifshitz say that the spin and orbital angular momentum are not independently conserved, it appears they are referring to the Dirac spin operator and the corresponding OAM, as evidenced by the fact that they refer to Σ as the spin operator for spin 1/2 particles in the next chapter (cf. Eq. 21.21).

In the abstract, we changed "the SAM-OAM splitting is unambiguous for massive particles" to "the angular momentum of massive particles has a natural splitting into the Wigner SAM and OAM operators". We have modified Sec. 3 (pp. 5-6) to include the explicit construction of the Wigner SAM operator for massive particles described above, and show that it commutes with the Hamiltonian. After this construction, we emphasize that this operator should not be confused with the Dirac SAM operator.

Referee comment: 2) I think that the last statement in the following sentences is too strong and should be changed.

"In particular, while the S_m operators commute with each other and thus generate an $\mathbb{R}^3$ symmetry, the S_p on the rhs of Equation (25) shows that the L_m do not form a Lie subalgebra. Thus, $\boldsymbol{L}$ does not generate any symmetry at all."

I agree that the L_m do not form a Lie subalgebra, but one can still exponentiate a given L_m to generate a symmetry operator. That is, since L_m is self-adjoint, $\exp(-i\theta L_m)$, for $\theta \in \mathbb{R}$, is still a unitary operator that maps photons to photons. One can build eigenstates of such operator, and I do not think that one can exclude that some material system could possess such a symmetry, that is, stay invariant after transformation with $\exp(-i\theta L_m)$.

As explained in Section 5 of Reference [12], since $[L_m, S_m] = 0$, one can see $\exp(-i\theta L_m)$ as the composition of a rotation and the transformation generated by S_m

Reply and Revisions:

We certainly agree with your statement, and this appears to just be an issue of the language we used. As you point out, any *single* self-adjoint linear operator A corresponds to a unitary

$\mathbb{R}$ symmetry via exponentiation (this symmetry may also sometimes descend to a symmetry of the circle group S^1 whose universal cover is $\mathbb{R}$). This corresponds to the fact that an operator always commutes with itself and thus spans the trivial 1D Lie algebra $\mathbb{R}$. And as you say, there does not appear to be any a priori reason to suggest that one cannot construct an interacting theory in which the Hamiltonian commutes with A so that A is conserved.

In this paper, we mean to focus on a different problem, namely the question of whether the *vector* operator $\boldsymbol{L} = (L_1, L_2, L_3)$ generates a symmetry for some Lie group G. Since we are concerned with the three-component vector operator $\boldsymbol{L}$ and not just a single component, G must be a three-dimensional Lie group (such as SO(3) in the case of $\boldsymbol{J}$ or $\mathbb{R}^3$ in the case of $\boldsymbol{S}$). The fact that the operators $\boldsymbol{L}$ do not form a closed Lie algebra shows that $\boldsymbol{L}$ does not generate such a symmetry. In the paper we used a boldfaced $\boldsymbol{L}$ to emphasize that we were referring to the vector operator rather than a single component. We have further emphasized this point by making two modifications. First, on p. 10, we have replaced the clause

> ...the S_p on the rhs of Equation (25) shows that the L_m do not form a Lie subalgebra.

with

> ...the S_p on the rhs of Equation (25) shows that the components of $\boldsymbol{L}$ do not combine together to form a Lie subalgebra.

Second, we have added a short paragraph at the end of the section (p. 11) reading:

> We note that while the vector operator $\boldsymbol{L}$ in the splitting (21)-(22) does not generate any 3D symmetry, any individual component, say L_3, commutes with itself. It therefore generates a 1D symmetry, and such symmetries can be of interest in interacting systems if they commute with the Hamiltonian, as discussed in Ref. [7].

Referee comment: 3) I find the following statement somewhat misleading:
"Massless fermions, known as Weyl fermions, are exceptionally rare, and have only been observed within the last decade in exotic materials."

It is my understanding that, for these quasi-particles, the linear dispersion relations that inspire the adjective "massless" do not have the same slope as a true massless particle in free space. In other words, the speed of light in such materials is smaller than c_0. Moreover, such dispersion relations are only approximately linear in the vicinity of a given point, and become more complicated when going away from such point.

Reply and Revisions:

We thank you for drawing our attention to this issue. As you point out, these quasiparticles have asymptotic behavior which is similar to massless particles, but that there are important differences between such quasiparticles and true massless elementary particles. As our formalism and results are concerned with elementary particles rather than such quasiparticles, we have removed reference to Weyl fermion quasiparticles. In particular, the original passage in Sec. II

> We restrict our discussion to non-projective representations of $\mathrm{ISO}^+(3,1)$, which corresponds to treating only particles with integer spin or helicity, i.e., bosons. Nearly all known massless particles are bosons, so this assumption has little effect on the generality of our results. Massless fermions, known as Weyl fermions, are exceptionally rare, and have only been observed within the last decade in exotic materials [8–10].

has been changed to

> We restrict our discussion to non-projective representations of $\mathrm{ISO}^+(3,1)$, which corresponds to treating only elementary particles with integer spin or helicity, i.e., bosons. All elementary massless particles in the Standard Model are bosons, so this assumption has little effect on the generality of our results.

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