Response to Report B on "Four no-go theorems on the existence of spin and orbital angular momentum of massless bosons"

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Dear Referee,

We thank you for reviewing our article and respond to your comments below.

Referee comment:

1. I disagree with this statement in the abstract: "***Moreover, it has been shown that most of the proposed SAM and OAM operators do not satisfy the defining commutation relations of angular momentum operators and are thus not legitimate splittings.***" Consequently, I partially agree with the first half of the statement since the relativistic gauge-dependent SAM and OAM operators fulfill the cyclic commutation relations, and the gauge-independent operators do not. However, the second half of the statement about the splitting being non-legitimate is not fully correct in my opinion. The reasons for my concern are covered in the comments listed below.

2. One of the primary concerns addressed in this paper revolves around the angular momentum operators for massless bosons that do not follow the SO(3) commutation relations. This has been highlighted in the Introduction, in the statement: "However, most of these operators have been shown to either not be well defined or else not actually satisfy SO(3) commutation relations and are therefore not genuine angular momentum operators." This directly questions the definition of Angular Momentum in the context of massless bosons. The fundamental definition of the angular momentum operator comes from the fact that it is a generator of rotation for the corresponding symmetry group under analysis. In DOI: 10.1103/PhysRevA.26.3428 (1982), Lenstra and Mandel showed that the total angular momentum operator generates rotations for a general massless field, which is also highlighted in DOI: 10.1364/JOSAB.524752 (2024). **However, the derivations done by Lenstra and Mandel in 1982 do not imply that the commutation relations from SO(3) group is a necessary rule for angular momentum operators for the massless field.**

Reply:

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The requirement that the total angular momentum operators $\boldsymbol{J}$ satisfies

$$[J_a, J_b] = i\epsilon_{abc}J_c \tag{1}$$

is a direct consequence of the Poincaré symmetry (Lorentz symmetry + spacetime translation symmetry) of the universe. One of the Wightman axioms of quantum field theory is that the Hilbert space of particle states is a unitary representation of the Poincaré group $\mathrm{ISO}^+(3,1)$ (see for example [1] Defn. 6.1, [2] Sec. 5.5 iii., or [3] p. 56, II.1.2 (A.1)). Indeed, since Wigner [4] first noted the correspondence between particles and irreducible unitary representations of the Poincaré group, essentially all formulations of quantum field theory assume that the single particle states (and by extension the multiparticle Fock space) have well-defined transformations under the Poincaré group i.e. are a representation of the Poincaré group. Any representation of the Lie group $\mathrm{ISO}^+(3,1)$ has an induced representation of the Lie algebra $\mathfrak{iso}(3,1)$. We will focus on the subgroup of Lorentz transformations $\mathrm{SO}^+(3,1)$ and its Lie algebra $\mathfrak{so}(3,1)$. The generators of any representation of $\mathfrak{so}(3,1)$ are the Hamiltonian H, the momentum operators $\boldsymbol{P}$, the (total) angular momentum operators $\boldsymbol{J}$, and the boost generators $\boldsymbol{K}$ which satisfy the commutation relations ([5], p. 61)

$$[J_a, J_b] = i\epsilon_{abc}J_c\,, \quad [J_a, K_b] = i\epsilon_{abc}K_c\,, \quad [K_a, K_b] = -i\epsilon_{abc}J_c \tag{2}$$

$$[J_a, P_b] = i\epsilon_{abc}P_c\,, \quad [K_a, P_b] = iH\delta_{ab}\,, \quad [K_a, H] = iP_a \tag{3}$$

$$[J_a, H] = [P_a, H] = [P_a, P_b] = [H, H] = 0. \tag{4}$$

In particular, there is a canonical copy of 3D rotations $\mathrm{SO}(3)$ in $\mathrm{SO}^+(3,1)$ (describing the observed rotationally symmetry of universe) and a corresponding canonical copy of $\mathfrak{so}(3)$ in $\mathfrak{iso}(3,1)$. The generators of this $\mathfrak{so}(3)$ subalgebra are the total angular momentum operators $\boldsymbol{J}$ and they satisfy (1) as seen from (2). This is an infinitesimal version of the requirement that acting on a state with a rotation $R_1 \in \mathrm{SO}(3)$ followed by another rotation $R_2 \in \mathrm{SO}(3)$ gives the same result as acting on the state with the the composite rotation R_2R_1. We emphasize that this is purely a mathematical consequence of the physical requirement that the theory is Poincaré symmetric. Indeed, their is a basis of the Lie algebra $\mathfrak{so}(3,1)$ which also satisfy Eqs. (2)-(4), and by the definition of a Lie algebra representation ([6], Defns. 16.12 and 16.36), the generators $(H, \boldsymbol{J}, \boldsymbol{K})$ must satisfy the same commutation relations as the Lie algebra. We also emphasize that everything above applies to both massive and massless quantum field theory. In the massive case, one can take the nonrelativistic limit, and one instead obtains a representation of the Galilean group which still has a canonical copy of 3D rotations SO(3). The generators of the corresponding Lie algebra representation are the nonrelativistic angular momentum operators and which must also satisfy Eq. (1) ([5], p. 62). Lorentz/Poincaré symmetry is a fundamental requirement of any relativistic theory, and it is for this reason that the the commutation relations Eq. (1) are assumed throughout the literature [5, 7–14]. Exceptions are the recent papers you mention by Yang *et al.* [15] and Das *et al.* [16]), which we will address in our reply to your subsequent comments.

We now address your reference to the article by Lenstra and Mandel [17] titled "Angular momentum of the quantized electromagnetic field with periodic boundary conditions". We point that since they quantize the electromagnetic field in a box of side length L with periodic boundary conditions. These boundary conditions explicitly break Poincaré symmetry (as also noticed by Van Enk and Neinhuis on p. 968 of Ref. [18]) so it is unclear how relevant their results are to the problem at hand. Regardless of this limitation, their results are concerned with the commutators of the fields with the angular momentum operators, not with the commutation relations between the angular momentum operators themselves. As seen from axiomatic treatments of QFT, the requirement that the Hilbert space is a unitary representation of the Poincaré group is separate from the requirement of how the fields transform under Poincaré transformations. Compare, for example, axioms A.1 and D in Haag [3] (pp. 56-57), or WA1 and WA6 in De Faria and De Melo [1] (pp. 118-119). This is to say, the relativistic invariance of the particle states requires Eq. (1) regardless of how the field commutes with angular momentum operators. We do remark, however, that the commutators of field operators with the angular momentum operators are always consistent with Eq. (1). Indeed, let $\phi(x)$ be some quantum field operator, $\Lambda \in \mathrm{ISO}^+(3,1)$, and Σ be the representation of $\mathrm{ISO}^+(3,1)$ on the particle states with corresponding Lie algebra representation η. Then the action on the fields is given by the adjoint action induced by Σ ([19], Eq. 11.67)

$$\mathrm{Ad}_\Lambda \phi(x) \doteq \Sigma_\Lambda \phi(x) \Sigma_{\Lambda^{-1}} \tag{5}$$

The induced Lie algebra representation of $\mathfrak{iso}(3,1)$ is the ad-representation where if $\omega \in \mathfrak{iso}(3,1)$ then

$$\mathrm{ad}_\omega \phi(x) = [\eta_\omega, \phi(x)]. \tag{6}$$

Restricting to $\omega \in \mathfrak{so}(3)$ we have a representation of $\mathfrak{so}(3)$ so in the standard basis

$\{\omega_1, \omega_2, \omega_3\}$ of $\mathfrak{so}(3)$, the ad-representation must obey the standard $\mathfrak{so}(3)$ commutation relations

$$[\mathrm{ad}_{\omega_a}, \mathrm{ad}_{\omega_b}] = i\epsilon_{abc}\mathrm{ad}_{\omega_c}. \tag{7}$$

Note that by definition $\eta_{\omega_a} = J_a$. Applying both sides of Eq. (7) to $\phi(x)$ and using Eq. (6) and the Jacobi identity gives

$$[[J_a, J_b], \phi(x)] = [i\epsilon_{abc}J_c, \phi(x)]. \tag{8}$$

This equation is satisfied if Eq. (1) holds. While in theory it may be possible to find some other commutation relation among the J_a which satisfy Eq. (8), we again emphasize that Eq. (1) is fundamentally required by the Poincaré symmetry of the particle states, not because of Eq. (8).

Referee Comment:

> 2. (cont.) Since the article deals with massless bosons, the little symmetry group ISO(2) should be the central theme for the angular momentum calculations, and not SO(3). The authors should consider modifying the sections to include the ISO(2) symmetry group, as opposed to SO(3) for these reasons.

Reply:

We first note that we do talk about the little group extensively throughout this paper, and its relationship to the obstruction of an SAM-OAM decomposition for massless particles. However, the little group is unrelated to the commutation relations that the total angular momentum operators $\boldsymbol{J}$ should satisfy. Indeed, the little group method [4, 5] was invented by Wigner as a systematic method to construct representations of the Poincaré group $\mathrm{ISO}^+(3,1)$ which correspond to particles. The little group is an important theoretical tool, but ultimately the constructed particle representation of $\mathrm{ISO}^+(3,1)$ contain a canonical copy of SO(3) whose generators are the total angular momentum operators [5], following from the same construction discussed in our reply to the previous comment. It is of particular note that this copy of SO(3) corresponds to the subgroup of rotations in both the massive and massless cases; while the little group does happen to be SO(3) in the case of massive particles, this is an altogether different copy of SO(3) [20]. The total angular momentum operators always correspond to this 3D rotational symmetry, regardless of the little group, and thus always satisfy Eq. (1). Eqs. (11) and (12) in the paper hold for any representation of the Poincaré group, we have made no assumptions other than the fundamental assumption that particle states are Poincaré symmetric, which is one of the foundational assumptions of a relativistic theory.

Referee Comment:

> 3. **Equations 11 and 12** needs modifications, since these equations are valid for gauge-dependent angular momentum operators, and not for the gauge-independent case(please refer Paragraph 2 of Point 2 above). In fact, Reference (16) from the paper and DOI: 10.1364/JOSAB.524752 (2024) proves this statement.

Reply:

Equations (11) and (12) in the paper impose the commutation relations $[J_a, J_b] = i\epsilon_{abc}J_c$ on the total angular momentum operator. As discussed in the reply to the first comment, this is required by the Poincaré symmetry of the theory. The total angular momentum operator for photons indeed satisfies this equation. This is derived, for example, in the Gupta-Bleuler formalism by Ref. [11] Eq. (27). Furthermore, this total angular momentum is gauge-invariant ([11], Sec 3.C).

In Ref. [16] you cite, Das *et al.* use a non-covariant quantization scheme to suggest that the total angular momentum operators do not satisfy Eq. (1) but instead

$$[\hat{J}_a, \hat{J}_b] = i\hbar\epsilon_{mnp}\hat{L}_c \tag{9}$$

where $\hat{\boldsymbol{L}}$ is another operator they define. This would indicate that the theory violates Lorentz symmetry, which would be at odds with the dictates of special relativity on which particle physics is based. To us, it seems that the most reasonable interpretation of Eq. (9) is that in this quantization scheme $\hat{\boldsymbol{J}}$ is not the correct total angular momentum operator. A similar conclusion was reached by Leader and Lorcé in 2019 ([10], pp. 23-24) when discussing previous proposed angular momentum operators. We quote, "...if other methods are utilized to construct new variants of the angular momentum operators and these fail to satisfy the commutation relations Eq. (1), that does not imply a 'sick' theory, but simply that the constructed operators are not genuine angular momentum operators."

We also emphasize that second quantization is not a single strictly defined procedure. It is rather a set of related procedures one applies to a classical field theory in the hopes of producing a well-behaved quantum field theory. There are points, particularly when quantizing gauge fields, where one must make choices on how to perform the quantization and there can be multiple quantization procedures which lead to satisfactory quantum field theories (e.g. Gupta-Bleuler vs. non-covariant quantization). It is always necessary to check at the end that one has in fact produced a well-behaved quantum field theory, and in particular that one has a genuine representation of the Poincaré group i.e. that the $\mathfrak{iso}(3,1)$ generators satisfy the commutation relations (2)-(4). Das *et al.* [16] use the Coulomb gauge which is not Lorentz invariant, and so it is easy to lose track of the Lorentz symmetry during quantization. There is no problem with utilizing such non-covariant quantization provided that in the end it produces a quantum field theory which is Poincaré invariant. The particular non-covariant quantization used in [16] is a common one and known to produce a valid quantum field theory, but it is not a natural setting to discuss problems which are explicitly related to Lorentz symmetry. In particular, it is not obvious how to write down angular momentum operators which satisfy the required commutation relations (1). The work of Das *et al.* [16] shows that $\hat{\boldsymbol{J}}$ defined by their Eq. (1) satisfies (9) rather than (2) and is thus not a satisfactory candidate for the total angular momentum operator by the standard definition.

We discuss the paper by Yang *et al.* [15] in our response to the next comment.

Referee Comment:

> 4. This line in section 3 "**The defining property of angular momentum operators is that they generate** SO(3) **symmetries**", which talks about massless bosons needs to be modified, since for massless bosons, the group ISO(2) should be the primary center of study, and not SO(3) which is the case for massive particles. This negates the need for reinforcing Equations 15 and 16. This directly leads to the modification of No-Go Theorem 1 in the context of massless bosons. The last line of Section 3 "The other set of operators, (L_M^{obs}, L_M^{obs}), are gauge invariant, but the $L_{M,i}^{obs}$ commute with each other rather than satisfy cyclic SO(3) relations and are therefore not angular momentum operators." should be

modified for the same reason.

Reply:

As discussed in the reply to the previous comment, the total angular momentum operators must satisfy SO(3) commutation relations as generators of a the subgroup of 3D rotations SO(3) of the Poincaré group. This is required by the assumption of Poincaré invariance central to modern particle physics, and is unrelated the little group or whether or not a particle is massive or massless. However, the term "angular momentum operators" have also been applied to other operators besides the total angular momentum operators, notably the SAM and OAM operators for massive particle in relativistic or nonrelativistic quantum mechanics. The standard definition in the literature [6, 7, 9–14, 21] is that generic angular momentum operators $\boldsymbol{L}$ are generators of an SO(3) representation, or equivalently, that they satisfy Eq. (1). There is good reason for this definition, as essentially all of the theoretical tools related to angular momentum operators rely on this assumption. For example, the addition of angular momentum formula ([6], Sec. 17.9), the Clebsch-Gordan formalism ([22], Sec. 15.2), the Casimir invariance of L^2 and the form of its eigenvalues as $\ell(\ell+1)$ ([6], Prop. 17.8), and the multiplet structure of the irreducible representations in terms of eigenstates of (L^2, L_z) and the theory of ladder operators ([22], Sec. 12.5) all follow from the assumption that $[L_a, L_b] = i\epsilon_{abc}L_c$. The purpose of this paper is to explore potential SAM-OAM decompositions using this standard definition.

To emphasize this, we have modified the passage you first quote to now read

> In this article, we take the defining property of angular momentum operators to be that they generate SO(3) symmetries, as is standard in the literature [6, 7, 9–14, 21]. This ensures that standard results about angular momentum operators hold such as the addition of angular momentum equation, the Clebsch-Gordan formalism, the Casmimir invariance of the S^2 and L^2, and the angular momentum multiplet structure of the eigenspaces of (L^2, L_z) and (S^2, S_z). We note that there has been some work in which this requirement on angular momentum operators is not imposed [15, 16] and that our results do not apply in such a setting.

We have also modified the second passage you quote, in which we discuss the operators defined by Yang *et al.* [15], to emphasize that our comments assume the standard definition of angular momentum operators as generators of an SO(3) representation

> The other set of operators, $(\boldsymbol{S}_M^{\text{obs}}, \boldsymbol{L}_M^{\text{obs}})$, are gauge invariant, but the $S_{M,i}^{\text{obs}}$ commute with each other rather than satisfy cyclic $\mathfrak{so}(3)$ relations, and are therefore not angular momentum operators by the standard definition.

Referee comment: 5. There are a couple typos in the paper:

(a) "Massless" in Section 3 title

(b) Some parts of the paper says SO(3), while others say $\mathfrak{so}(3)$

Reply:

(a) Thank you for catching this.

(b) SO(3) and $\mathfrak{so}(3)$ are different objects, the former is the Lie group and the latter is the corresponding Lie algebra. The use of capital letters for Lie groups and Goethic/Fraktur font for the corresponding Lie algebras is standard notation, see for example Refs. [23] p. 211, [24] p. 230, [2] p. 9, or [25]. Our first use of this notation was on page 4, in which we wrote

> The Lie group representations Σ and $\tilde{\Sigma}$ of $\text{ISO}^+(3,1)$ induce corresponding Lie algebra representations η and $\tilde{\eta}$ of $\mathfrak{iso}(3,1)$ describing the infinitesimal group actions ...

which we believe makes our notation unambiguous.

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[20] In fact, there is no single copy of SO(3) in the Lorentz group which corresponds to the little group of massive particles. At each $\boldsymbol{k}$ the little group is a different copy of SO(3) embedded in the Lorentz group.

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