
Responses to Referee 2:

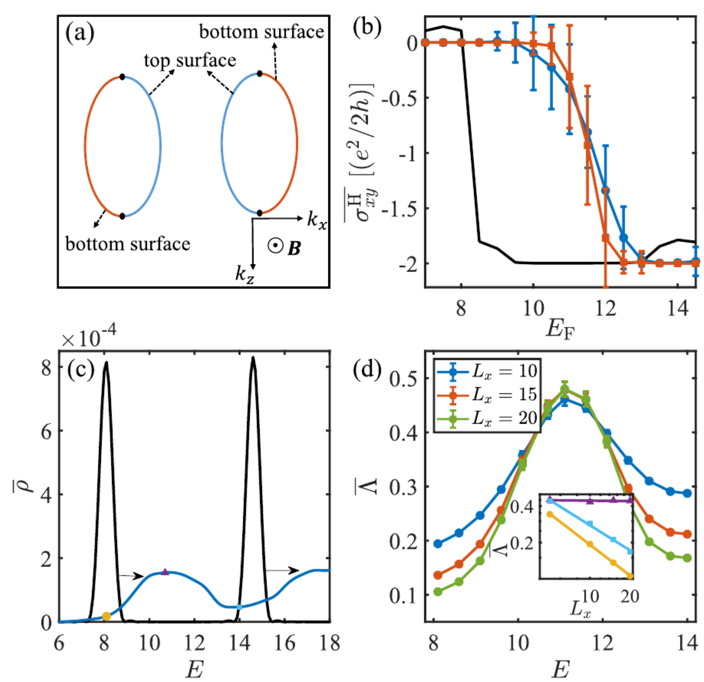
We thank the referee for the careful reading of our manuscript and the recommendation for publication given the comments are addressed based on the positive judgement that our paper is "clearly written with interesting results", providing a "novel way of detecting topology in amorphous topological metals". We also appreciate his/her constructive suggestions/questions which are invaluable in improving our paper. The following contains our detailed response to specific points.

Comment 1: - potentially novel way of detecting topology in amorphous topological metals
- clearly written manuscript

Reply: We thank the referee for the high evaluation of our paper.

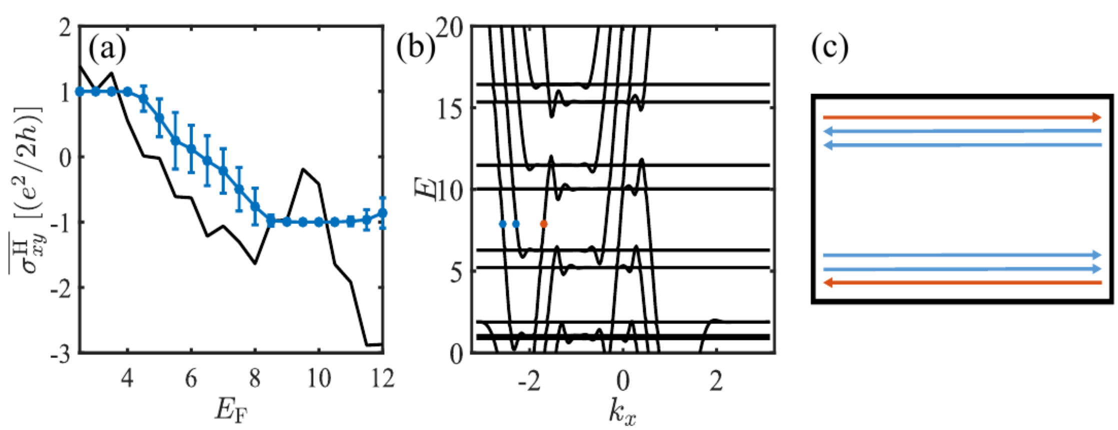
Comment 2: the more interesting case of topological semimetal phase protected by the time reversal symmetry is less explored.

Reply: We thank the referee for the nice suggestion. In the original version, for the time-reversal symmetric case we did not provide numerical results as rich as the case without the time-reversal symmetry, because the time-reversal symmetric case consumes much more computational resources due to the doubled internal degrees of freedom and we did not expect qualitatively different results. In the revised version, we have followed the referee's suggestion to calculate the Bott index, density of states and localization length of the time-reversal invariant case, as shown in Sec. 6 and Fig. 5 of the revised version, with Fig. 5 reproduced in the following.



We find that the Bott index on amorphous lattices [see the red line in the above figure (b)] agrees well with the Hall conductance (blue line). Similar to the case without time-reversal symmetry, the shift of the plateau of the Hall conductance in energy is attributed to the right-moving behavior of the Landau levels as shown in the above figure (c) with arrows indicating the shifting direction of Landau levels induced by structural disorder. Although the Landau levels overlap with each other and renders the system gapless, the bulk states are localized except at a critical point, demonstrated by the normalized localization length in the figure (d). We see that the normalized localization length decreases as L_x increases far away from $E = 10.7$ and tends to converge to a constant near this energy. The inset shows the result of the finite-size scaling calculations. For $E = 8.1$ and 14, we observe a finite slope, indicating the localization behavior, whereas for $E = 10.7$, we see a vanishing slope, which is a signature of the critical extended property. Therefore, the quantized Hall conductance arises from the edge states that remain extended; meanwhile, the localized bulk states do not contribute to the transport. The generic mechanism indicates that 3D quantum Hall effect may be broadly used to detect the topology of time-reversal symmetric topological semimetals, provided that the Landau levels are not too close in the energy space.

We have also studied the time-reversal invariant case under a tilted magnetic field with $\theta = 10°$. Surprisingly, on amorphous lattices we see a transition from a plateau of Hall conductance at $e^2/(2h)$ to a plateau at $-e^2/(2h)$ as shown in the following figure (a) (the blue line), in stark contrast to the case under a vertical magnetic field with Hall conductance quantized at even integer values in units of $e^2/(2h)$, shown in Fig. 5(b). However, on a regular lattice, no quantized plateaus of Hall conductance arise in the same energy region up to $L_x = 550$ and no signature of quantization is observed during the increasing of the system size. To elaborate on the reason why quantized Hall conductance emerge on amorphous lattices while is absent on the regular lattice, we show the energy spectrum with respect to the lattice momentum k_x on a regular lattice in the following figure (b), exhibiting Landau levels with edge states going upwards and downwards alternately. For example, within the gap near $E = 8$ (indicated by the blue and red points), there exist two edge states propagating in the same direction (indicated by blue points) and one in opposite direction (indicated by the red point) on each edge, also shown schematically in the following figure (c). The quantized Hall conductance is prohibited by the backscattering between the counter-propagating edge states on each edge. On amorphous lattices, a pair of counter-propagating edge states become localized due to disorder with only one edge channel left, resulting in the quantized Hall conductance.



We have added the figures and the corresponding discussion in the Sec. 6 and Sec. 7 in the revised manuscript.

Comment 3: it is not clear whether the results are generic to all time-reversal symmetry protected topological phases.

Reply: We thank the referee for the question. In topological semimetals with Fermi arcs on the surfaces and Weyl or Dirac points in the bulk, the 3D quantum Hall effect arises from the Weyl orbit. For gapped topological phases such as 3D topological insulators with time-reversal symmetry, while Weyl orbits do not exist under magnetic fields, the Dirac cone surface states can still lead to the appearance of Landau levels under magnetic fields, resulting in the quantum Hall effect. Recent experiments have observed spin-momentum locked surfaces in amorphous Bi_2Se_3 (Ref. [72]). In the presence of a magnetic field, these surface states may contribute to the formation of Landau levels, as suggested by our results that even in completely random lattices, Landau levels can exist, albeit broadened, and can still give rise to quantized Hall conductance. Thus, quantum Hall effects in amorphous Bi_2Se_3 deserves to be studied both theoretically and experimentally.

We have added a sentence "In addition, our results demonstrate the presence of Landau levels in 3D amorphous systems, which may stimulate further investigation into the quantum Hall effect in amorphous topological insulators [72], such as Bi_2Se_3, under magnetic fields." in the revised manuscript.

Comment 4: The authors show that the amorphous topological metallic systems may exhibit a 3D quantum Hall effect in presence of the magnetic field. Since this phenomenon can be observed experimentally, the work offers a way to detect/confirm topological features in amorphous metallic systems that do not admit momentum-space topological invariants. This is particularly relevant for Weyl semimetals protected by the time-reversal symmetry for which most of the real-space approaches to calculating the topological invariant fail, with the exception of the spectral localizer (see Schulz-Baldes & Stoiber EPL 136 27001 (2021) and J. Math. Phys. 64, 081901 (2023); Cerjan & Loring PRB 106 064109 (2022); Dixon et al. PRL 131 213801 (2023); Franca & Grushin arXiv: 2306.17117).

Reply: We thank the referee for the nice summary of our paper and for judging our work "particularly relevant" for the study of Weyl semimetals. We also thank the referee for bringing us the spectral localizer with references and we have added the references in the introduction of the revised manuscript.

Comment 5: I find the manuscript clearly written with interesting results. Provided the authors answer my comments, I would be happy to recommend this work for publication in Scipost Physics Core.

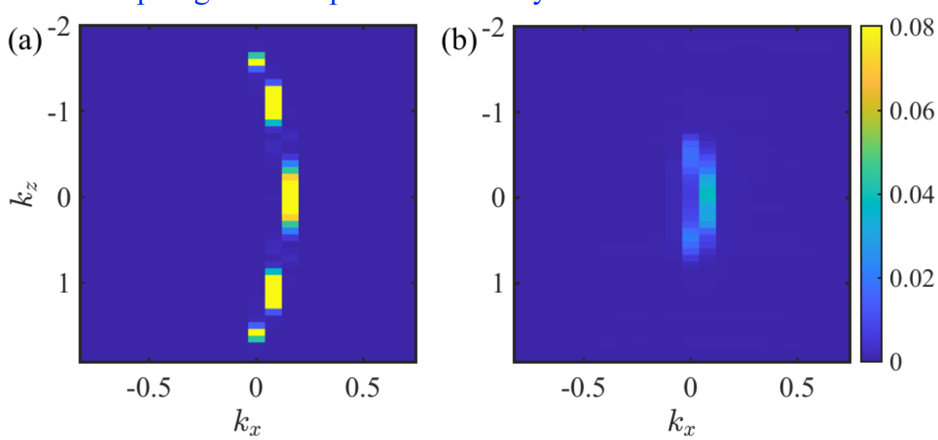
Reply: We thank the referee for the high evaluation of our paper and his/her recommendation for publication after we address the comments. Based on the referee's questions and suggestions, we have carried out substantial new calculations and significantly revised the manuscript and hope that this strengthened manuscript will convince him/her to recommend publication of this work in SciPost Physics rather than SciPost Physics Core. In the following, we would like to clarify why our paper is suitable for publication in SciPost Physics.

First, we show that the 3D quantum Hall effect can serve as a probe of the Weyl-band-like topology in amorphous systems, especially in the case with time-reversal symmetry that the topology is hard to detect by other methods. The amorphous materials are ubiquitous in nature, and the breakthrough of observing topological surface states in amorphous Bi_2Se_3 by ARPES will definitely stimulate intense interest in the search of more amorphous topological materials, including amorphous topological semimetals which have never been found. Thus, our findings providing new ways to detect topology in amorphous systems are highly relevant to the current research, thereby significantly advancing the field of amorphous Weyl/Dirac metals.

Second, the 3D quantum Hall effect in crystalline topological semimetal has been experimentally observed in Cd_3As_2 [Nature 565 331 (2019)] and an important further step is to study the effect in non-crystalline materials as disorder plays a central role in physics and non-crystalline materials are ideal platforms to study the effect of disorder in real materials. Therefore, our paper is highly relevant to the ongoing experimental development. We have slightly modified the conclusions in the revised manuscript accordingly.

Comment 6: I do not agree with the authors that ARPES cannot be used to probe amorphous systems as it was in fact used in their Ref. 70 to demonstrate experimentally the existence of a topological phase in amorphous Bi2Se3. This is because in amorphous systems, we can still have well defined plane waves of momentum k describing the outgoing electron. The ARPES measures the overlap of these plane waves with the eigenstates of the Hamiltonian, implying that sharp spectral features in ARPES of amorphous systems indicate the presence of states.

Reply: We thank the referee for the important comment. We have followed Ref. [70] (Ref. [72] in the current version) to calculate the surface spectral function on both regular and amorphous lattices as shown in the following figure. We see clear Fermi arcs on the regular lattice [figure (a)] but vague signature on amorphous lattices [figure (b)], suggesting that the ARPES measurement of the topological amorphous metal may be difficult.



For clarity, we have replaced the term "impractical" with "difficult" in the introduction of the revised manuscript. We have also added the figure and corresponding discussion in Appendix A of the revised manuscript.

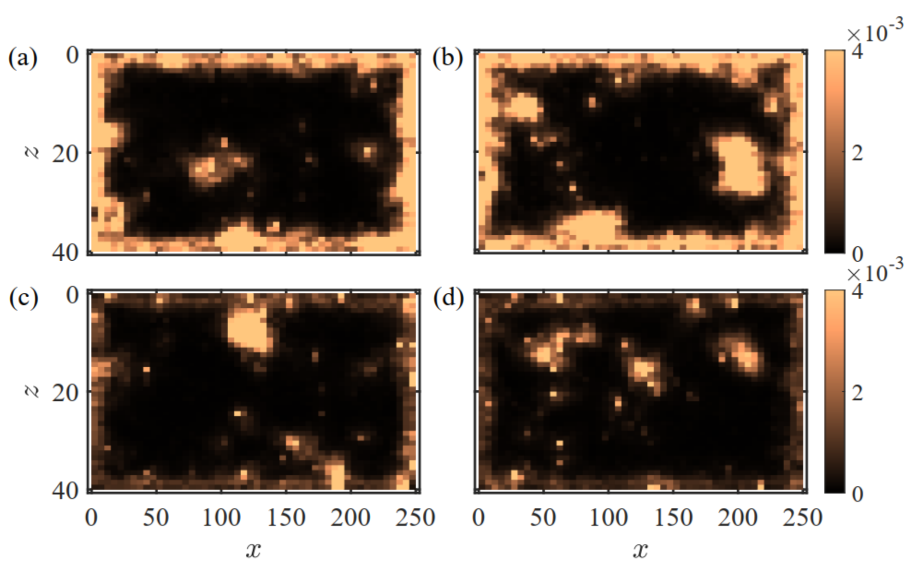
Comment 7: In the last line of page 3, the authors set a value for parameter A that I could not find previously defined in the text. Since their Ref. 8 uses a parameter A in the Hamiltonian, and they use γ, I wonder shouldn't A be replaced with γ? This should also explain why they never set the value of parameter γ in the manuscript.

Reply: We thank the referee for pointing out the typo. We have followed the referee's suggestion to replace A with γ in the revised manuscript.

Comment 8: In Figs. 2(c) and (d), it is very unusual to see that the bulk states remain so visible even after averaging over 100 configurations. Could the authors explain this?

Reply: We thank the referee for the question. We think that the high local density of states (LDOS) in certain locations in the bulk arise due to the finite number of samples. If we have infinitely many samples, the islands in the bulk with high values in the LDOS profile will be smoothed out. However, we only calculate 100 samples here due to the large consumption of computational resources, which are not enough to see a uniform LDOS distribution in the bulk.

We know that in many previous works on amorphous topological phases, a hundred of samples are enough for a smooth LDOS profile in the bulk. However, in our case, the fluctuations of LDOS over samples are much severe, requiring more samples. To show the large fluctuations, we display the LDOS of four samples in the following. We can identify edge states in all the four samples under the identical color map, but the edge states in (a) and (b) are much clearer than (c) and (d), implying large fluctuations of LDOS over the samples. We see localized bulk states with high LDOS in the samples, which is even much more visible than the edge states in (c) and (d), implying the large number of samples needed for a smooth distribution in the bulk. As the bulk states emerge at different positions over the samples, we deduce a uniform probability distribution for the occurrence of the bulk states over the spatial positions in the bulk and thus expect a smooth profile of bulk LDOS for an infinite number of samples. The large fluctuations are a specific property of our model, perhaps due to the presence of the magnetic field and the 3D nature. We have clarified this in Sec. 3 in the revised manuscript.



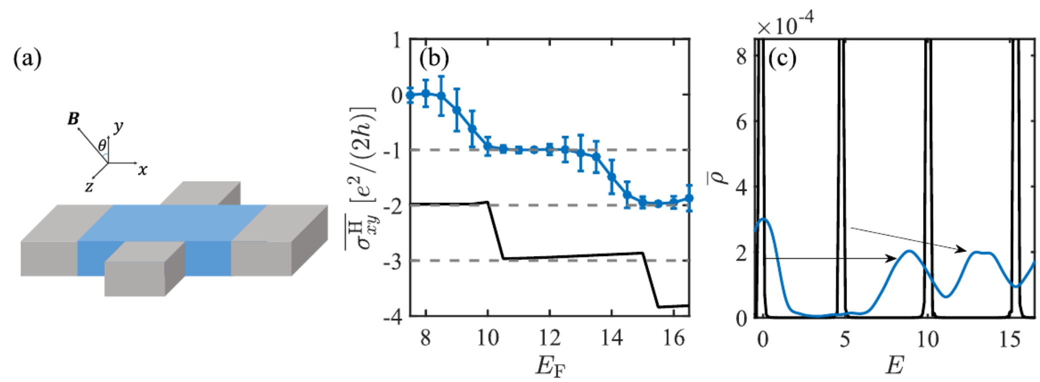
Comment 9: Concerning section 5 that focuses on the time-reversal symmetry protected Weyl semimetals, it is not clear to me whether the results the authors obtain are specific to this model

or are generic to this class of systems. It would significantly add to the value of the manuscript if the authors could provide a discussion on this.

Reply: We thank the referee for the question. We believe that the results are generic to time-reversal symmetric topological semimetals beyond the specific model we study. In the main manuscript, we show that the 3D quantum Hall effect can exist in amorphous systems because the bulk states are localized except at the critical point although the Landau levels broaden and overlap. The Chern number (Bott index) is carried by the critical states. Since this mechanism is generic, the quantum Hall effect should be able to exist in various models and a large range of system parameters as long as the separation between the Landau levels in the energy space is not too small. We have added a discussion on this point in Sec. 6 of the revised manuscript.

Comment 10: Fig. 6(b), we see that the Hall conductances for crystalline and amorphous systems have a very similar dependence on E_F, in comparison with the time-reversal broken case. What would be the reason for this? In addition, I find that having a calculation similar to Fig. 3(a) would enrich the manuscript.

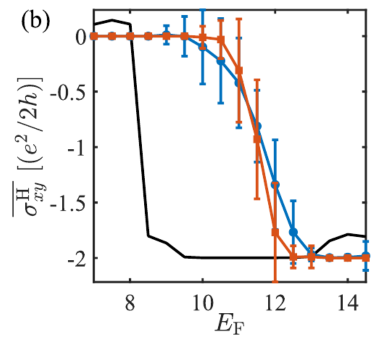
Reply: We thank the referee for the careful reading of our manuscript. We have followed the referee's nice suggestion to calculate the local density of states in this case and plot it in the following figure (c) along with the Hall conductance in Fig. 6. We see that in amorphous lattices, the Landau levels shift roughly the distance of two Landau levels compared with the regular lattice, indicated by the arrows. This explains why the dependence of Hall conductance on E_{F} looks similar except for a shift in vertical directions by $2e^2/(2h)$.



We have added the figure and the corresponding discussion in Sec. 7 of the revised manuscript.

Comment 11: Could the authors calculate the Bott index for Hamiltonian Eq. (6) in presence of magnetic field? It would be a great to show the bulk-boundary correspondence works for this case.

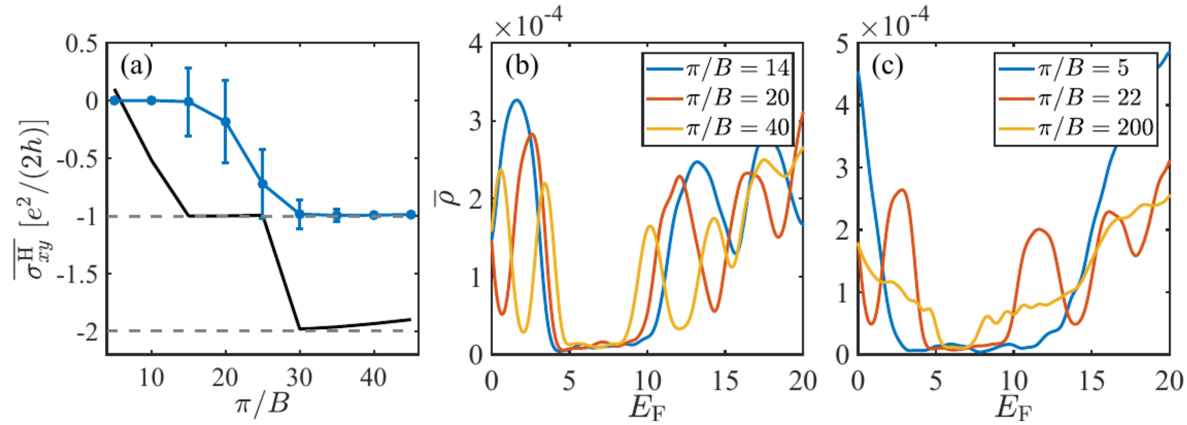
Reply: We thank the referee for the nice suggestion. We have followed the referee's suggestion to calculate the Bott index for the time-reversal symmetric case as shown in the following figure. The Bott index (red line) agrees well with the Hall conductance (blue line), revealing the bulk-boundary correspondence in the system.



We have added the figure and the corresponding discussion in Sec. 6 of the revised manuscript.

Comment 12: Finally, I wonder whether about the interplay between disorder strength and the magnetic field. In the presence of disorder, does the nonzero Hall conductance appear for any magnetic field strength?

Reply: We thank the referee for the nice question. To address the question, we have calculated the Hall conductance with respect to the inverse of the magnetic field B at a fixed E_F, as shown in the following figure (also see Fig. 4 in the revised manuscript).



We see quantized plateaus in both regular (black line) and amorphous lattices (blue line) and as the magnitude of the magnetic field decreases (π/B increases), the Hall conductance transitions from one plateau to another one with larger absolute value. The transition is attributed to the shifting of the Landau levels to lower energy as the magnetic field decreases, as shown in the above figure (b) for amorphous lattices. We also observe that the plateaus of the Hall conductance on amorphous lattices float upwards compared with the regular lattice, which is caused by the right moving of the Landau levels in the energy space induced by the structural disorder as we have explained in Fig. 3(a).

The magnitude of the magnetic field cannot be too small for the occurrence of 3D quantum Hall effect which can be understood easily by taking the limit of no magnetic field under which no quantum Hall effect arises. When the magnetic field is too large, the 3D quantum Hall effect will also disappear. Intuitively, as the strength of the magnetic field increases, the bulk states with energy further away from the Weyl points will be squeezed into Landau levels together with the

Fermi arc states near the Weyl points, because the degeneracy of the Landau level is proportional to the magnitude of the magnetic field. Away from the Weyl points, the bulk DOS is large, reminiscent of the conventional 3D metal, leading to the disappearance of well-separated Landau levels even on the regular lattice. Thus, the magnitude of the magnetic field has an upper bound for the 3D quantum Hall effect. To justify our arguments above, we calculate the DOS on amorphous lattices as shown in the above figure (c) under magnetic fields with magnitude $\frac{\pi}{B} = 5, 22, 200$. Only for moderate magnitude $\frac{\pi}{B} = 22$ (the red line), we see clear patterns of broadened Landau levels with stark contrast between peaks and valleys, which often implies the existence of the quantum Hall effect. Under too large (the blue line) and too tiny (the yellow line) magnetic field, Landau levels cannot be identified, indicating the absence of the quantum Hall effect. Therefore, the occurrence of the 3D quantum Hall effect requires the magnetic field in a moderate range of magnitude.

We have added a new Section 5 to discuss the effects of the magnitude of the magnetic field on the quantum Hall effect in the revised manuscript.