
Responses to Referee 1:

We thank the referee for the careful reading of our manuscript and the positive comments that our paper "has sufficient novelty and relevance to the scientific community", "quite relevant for current research in the field", "clear writing and presentation" and thus "warrant publication". We also appreciate his/her constructive suggestions/questions which are invaluable in improving our paper. The following contains our detailed response to specific points.

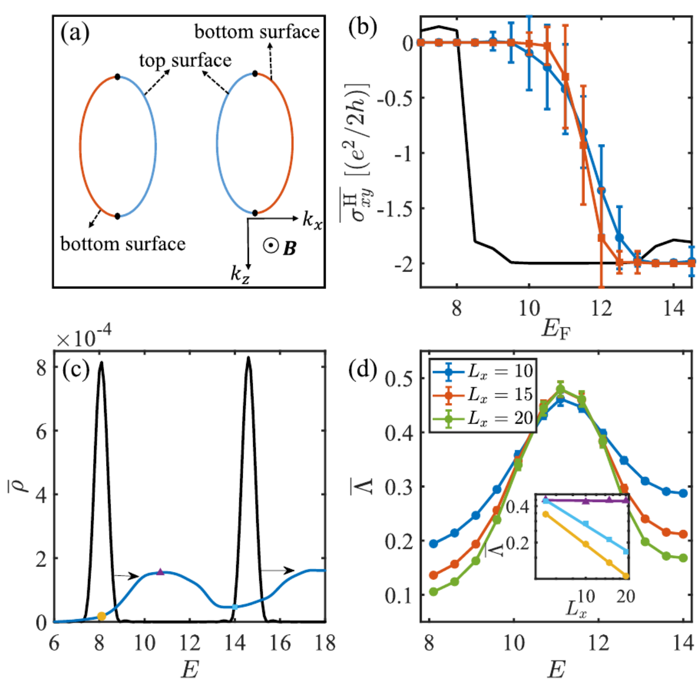
Comment:

1) Novel findings, with an approach motivated by experimental relevance.
2) Multiple methods applied to verify and explain results.
3) Generally clear writing and presentation.

Reply: We thank the referee for the high evaluation of our paper.

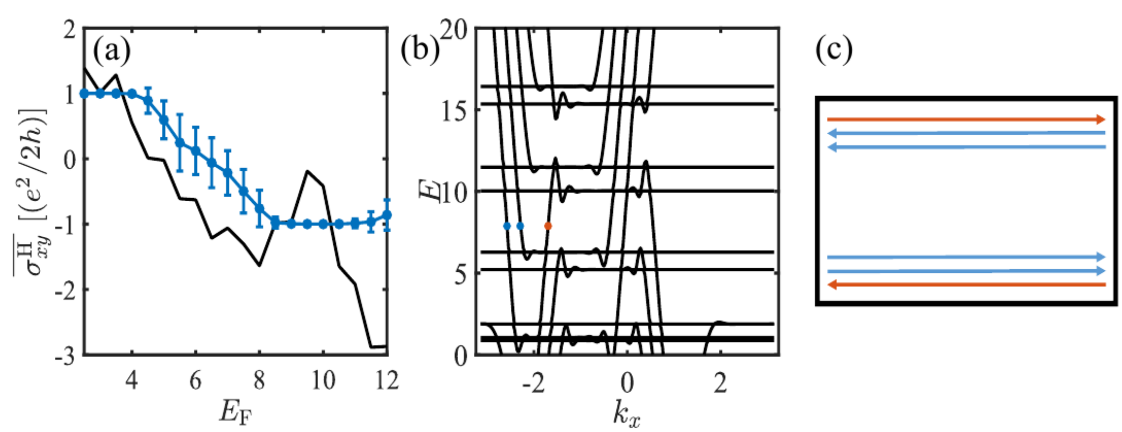
Comment 1: The cases with and without time-reversal symmetry could be compared and contrasted more; results are often obtained only for one or the other with no explanation for the choice.

Reply: We thank the referee for the nice suggestion. In the original version, for the time-reversal symmetric case we did not provide numerical results as rich as the case without the time-reversal symmetry, because the time-reversal symmetric case consumes much more computational resources due to the doubled internal degrees of freedom and we did not expect qualitatively different results. In the revised version, we have followed the referee's suggestion to calculate the Bott index, density of states and localization length of the time-reversal invariant case, as shown in Sec. 6 and Fig. 5 of the revised version, with Fig. 5 reproduced in the following.



We find that the Bott index on amorphous lattices [see the red line in the above figure (b)] agrees well with the Hall conductance (blue line). Similar to the case without time-reversal symmetry, the shift of the plateau of the Hall conductance in energy is attributed to the right-moving behavior of the Landau levels as shown in the above figure (c) with arrows indicating the shifting direction of Landau levels induced by structural disorder. Although the Landau levels overlap with each other and renders the system gapless, the bulk states are localized except at a critical point, demonstrated by the normalized localization length in the figure (d). We see that the normalized localization length decreases as L_x increases far away from $E = 10.7$ and tends to converge to a constant near this energy. The inset shows the result of the finite-size scaling calculations. For $E = 8.1$ and 14, we observe a finite slope, indicating the localization behavior, whereas for $E = 10.7$, we see a vanishing slope, which is a signature of the critical extended property. Therefore, the quantized Hall conductance arises from the edge states that remain extended; meanwhile, the localized bulk states do not contribute to the transport. The generic mechanism indicates that 3D quantum Hall effect may be broadly used to detect the topology of time-reversal symmetric topological semimetals, provided that the Landau levels are not too close in the energy space.

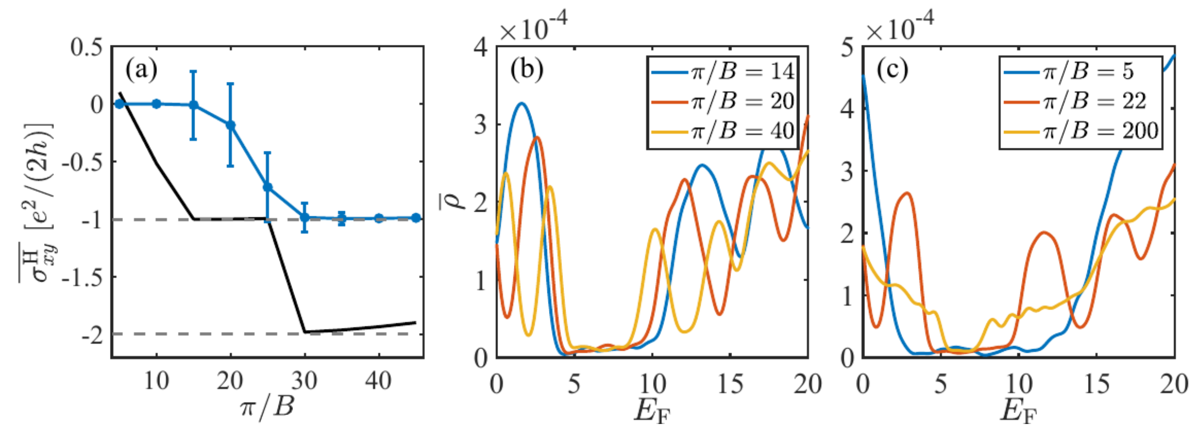
We have also studied the time-reversal invariant case under a tilted magnetic field with $\theta = 10°$. Surprisingly, on amorphous lattices we see a transition from a plateau of Hall conductance at $e^2/(2h)$ to a plateau at $-e^2/(2h)$ as shown in the following figure (a) (the blue line), in stark contrast to the case under a vertical magnetic field with Hall conductance quantized at even integer values in units of $e^2/(2h)$, shown in Fig. 5(b). However, on a regular lattice, no quantized plateaus of Hall conductance arise in the same energy region up to $L_x = 550$ and no signature of quantization is observed during the increasing of the system size. To elaborate on the reason why quantized Hall conductance emerge on amorphous lattices while is absent on the regular lattice, we show the energy spectrum with respect to the lattice momentum k_x on a regular lattice in the following figure (b), exhibiting Landau levels with edge states going upwards and downwards alternately. For example, within the gap near $E = 8$ (indicated by the blue and red points), there exist two edge states propagating in the same direction (indicated by blue points) and one in opposite direction (indicated by the red point) on each edge, also shown schematically in the following figure (c). The quantized Hall conductance is prohibited by the backscattering between the counter-propagating edge states on each edge. On amorphous lattices, a pair of counter-propagating edge states become localized due to disorder with only one edge channel left, resulting in the quantized Hall conductance.



We have added the figures and the corresponding discussion in the Sec. 6 and Sec. 7 in the revised manuscript.

Comment 2: Potentially relevant aspects regarding parameters, in particular the magnitude of the magnetic field, left vague or unexplored.

Reply: We thank the referee for the nice suggestion. To elaborate on the effect of the magnitude of the magnetic field on the 3D quantum Hall effect, we have followed the referee's suggestion to calculate the Hall conductance with respect to the inverse of the magnetic field B at a fixed E_F, as shown in the following figure (also see Fig. 4 in the revised manuscript).



We see quantized plateaus in both regular (black line) and amorphous lattices (blue line) and as the magnitude of the magnetic field decreases (π/B increases), the Hall conductance transitions from one plateau to another one with larger absolute value. The transition is attributed to the shifting of the Landau levels to lower energy as the magnetic field decreases, as shown in the above figure (b) for amorphous lattices. We also observe that the plateaus of the Hall conductance on amorphous lattices float upwards compared with the regular lattice, which is caused by the right moving of the Landau levels in the energy space induced by the structural disorder as we have explained in Fig. 3(a).

The magnitude of the magnetic field cannot be too small for the occurrence of 3D quantum Hall effect which can be understood easily by taking the limit of no magnetic field under which no quantum Hall effect arises. When the magnetic field is too large, the 3D quantum Hall effect will also disappear. Intuitively, as the strength of the magnetic field increases, the bulk states with energy further away from the Weyl points will be squeezed into Landau levels together with the Fermi arc states near the Weyl points, because the degeneracy of the Landau level is proportional to the magnitude of the magnetic field. Away from the Weyl points, the bulk DOS is large, reminiscent of the conventional 3D metal, leading to the disappearance of well-separated Landau levels even on the regular lattice. Thus, the magnitude of the magnetic field has an upper bound for the 3D quantum Hall effect. To justify our arguments above, we calculate the DOS on amorphous lattices as shown in the above figure (c) under magnetic fields with magnitude $\frac{\pi}{B} = 5, 22, 200$. Only for moderate magnitude $\frac{\pi}{B} = 22$ (the red line), we see clear patterns of broadened Landau levels with stark contrast between peaks and valleys, which often implies the existence of the quantum Hall effect. Under too large (the blue line) and too tiny (the yellow line) magnetic field, Landau levels cannot be identified, indicating the absence of the quantum Hall effect. Therefore, the occurrence of the 3D quantum Hall effect requires the magnetic field in a moderate range of magnitude.

We have added a new Section 5 to discuss the effects of the magnitude of the magnetic field on the quantum Hall effect in the revised manuscript.

Comment 3: The main finding of the work appears to be that the presence of magnetic fields can allow for a detection of Weyl bands in amorphous materials even when time-reversal invariance (in the absence of a magnetic field) causes the conductance from those bands to ordinarily cancel out. With recent experiments finding evidence for a 3D quantum Hall effects in crystalline system, and the challenges with measuring amorphous anomalous Hall conductance mentioned in the manuscript's introduction, the findings are quite relevant for current research in the field. The authors find that topology (as calculated by Bott index) and observables (Hall conductivity) match well, and explain the well-defined quantization by considering the localization length of bulk wavefunctions.

Reply: We thank the referee for the nice summary of our paper and judging our paper as "quite relevant for current research".

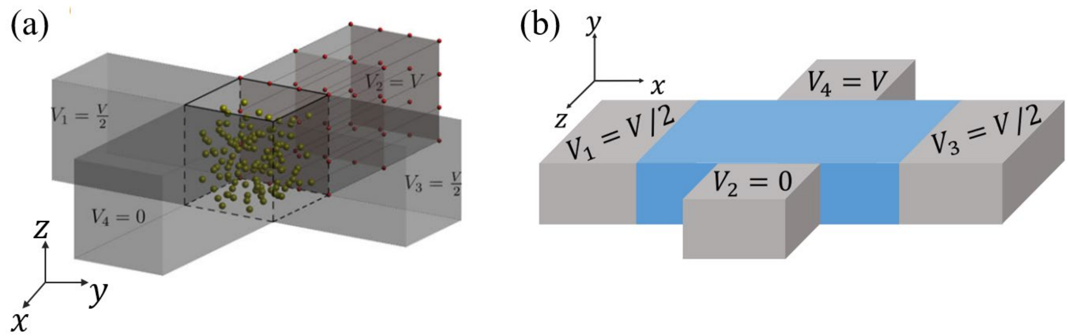
Comment 4: Two Hamiltonians are considered, first without and later with time-reversal symmetry. The latter seems potentially more significant as a finding, as the case with broken TRS, although without a magnetic field, was already considered in Ref. [54].

However, many of the results, e.g. the localization lengths and scaling, as well as the tilted magnetic field in the appendix, are only obtained for the TR-breaking case. Even if I do find it plausible the results would not be too dissimilar, an explicit discussion of this is absolutely warranted.

Reply: We thank the referee for the comment and the suggestion. We have followed the referee's suggestion to calculate the Bott index, density of states and localization length for the time-reversal invariant case (see our reply to Comment 1 and the added discussion in Sec. 6 in the revised manuscript) and study the case under a tilted magnetic field (see our reply to Comment 1 and the added discussion in Sec. 7 in the revised manuscript).

Comment 5: As for the TR-breaking case, given that [54] found a quantized Hall conductance with no external magnetic field, the claim that parameters are chosen "with no loss of generality" is not obvious to me. Intuitively, while the Hamiltonian is different, could not a large enough TR-breaking term even here yield the observed results regardless of any external magnetic field? An elaboration on the effects of the magnitude of B on the observed results would do much to clarify this. This would also add scientific novelty to this part (in taking more of distance from [54]).

Reply: We thank the referee for the question. We would like to clarify that the anomalous Hall conductivity found in Ref. [54] (Ref. [56] in this version) originates from the Fermi arcs (surface states) rather than Landau levels. There, the Hall conductivity is not quantized (see Fig. 2(a) in Ref. [56] in this version). In contrast, in our case, the quantized Hall conductance arises from the Landau level of the surface states. In the absence of magnetic fields, Landau levels do not exist so that the Hall conductance based on the terminal configuration shown in the following figure (b) vanishes.



To demonstrate this, we reproduce the terminal configuration in Ref. [54] (Ref. [56] in this version) in the above figure (a) for comparison. The Weyl points are separated in the z direction, i.e. located at $\boldsymbol{k} = (0,0,\pm k_w)$, so the four surfaces connected to terminals all have Fermi arcs, which contribute to the anomalous Hall conductance, which is not quantized. However, in our paper where the Weyl points are also separated in the z direction, the surface connected to terminal 2 and 4 do not have Fermi arcs due to the different terminal configuration, as shown in Fig.(b) in the following (note the difference in the coordinate system). Therefore, in the absence of a magnetic field, no conventional anomalous Hall conductivity is observed. Only under the magnetic field, the Fermi arcs on the top and bottom surfaces form Weyl orbits giving rise to 3D quantum Hall effect, which is explained in the manuscript. In summary, although our current paper and Ref. [54] both study the TRS-broken Weyl semimetal, the Hall conductance arise from completely different origins. Thus, Ref. [54] should not weaken the significance of our paper. We have given a brief discussion on this point in Sec. 3 in the revised manuscript.

In addition, we have shown that the 3D quantum Hall effect can exist in amorphous systems because the bulk states are localized except at the critical point although the Landau levels broaden and overlap. This mechanism is generic, so the quantum Hall effect can exist in a large range of system parameters as long as the distance between the Landau levels in the energy space is not too close. We have added a sentence on this point in Sec. 4 in the revised manuscript.

As in the reply to a comment above, we have also elaborated on the effect of the magnitude of the magnetic field following the referee's suggestion, as shown in Sec.5 in the revised version.

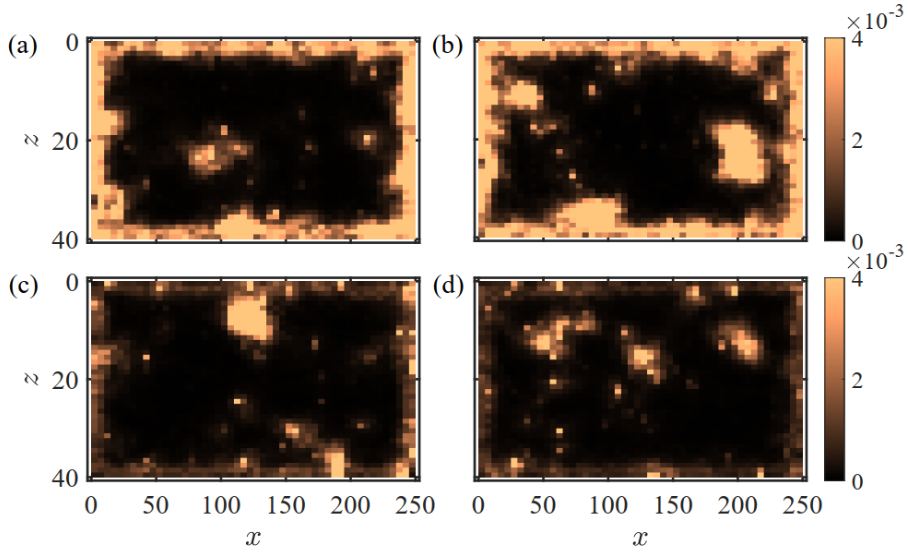
Comment 6: There are also some other details that could, in my opinion, improve the manuscript:
- It is specified that the sample used be thin along y. Whether this was a choice made for computational expedience, or something that is significant to the results, could be mentioned.

Reply: We thank the referee for the question. The sample cannot be too thick along the y direction for the 3D quantum Hall effect, because the Landau levels will become closer as the sample gets thicker. For an infinitely thick sample, the Landau levels will touch each other and no quantized Hall conductance can be observed. Note that the requirement for finite thickness is not only necessary for our amorphous system, but also for the pristine regular Weyl semimetal [PRL 119 136806 (2017), Nature 565 331 (2019)]. We have added a sentence on this issue in Sec. 2 of the revised manuscript and detailed analysis in Appendix B.

Comment 7: The LDOS figures show high values in certain locations in the bulk of the system. While this would be expected for individual realizations, it is not immediately obvious why this feature is seen in a sample-averaged quantity, where fluctuations due to random sites should be equally distributed over the bulk. Is it a finite-size effect due to the location of the leads? This could also be mentioned in the text.

Reply: We thank the referee for the question. We think that the high local density of states (LDOS) in certain locations in the bulk arise due to the finite number of samples. If we have infinitely many samples, the islands in the bulk with high values in the LDOS profile will be smoothed out. However, we only calculate 100 samples here due to the large consumption of computational resources, which are not enough to see a uniform LDOS distribution in the bulk. In addition, in our calculation of the LDOS, since we do not take into account of the leads, the phenomenon does not arise from their location.

We know that in many previous works on amorphous topological phases, a hundred of samples are enough for a smooth LDOS profile in the bulk. However, in our case, the fluctuations of LDOS over samples are much severe, requiring more samples. To show the large fluctuations, we display the LDOS of four samples in the following. We can identify edge states in all the four samples under the identical color map, but the edge states in (a) and (b) are much clearer than (c) and (d), implying large fluctuations of LDOS over the samples. We see localized bulk states with high LDOS in the samples, which is even much more visible than the edge states in (c) and (d), implying the large number of samples needed for a smooth distribution in the bulk. As the bulk states emerge at different positions over the samples, we deduce a uniform probability distribution for the occurrence of the bulk states over the spatial positions in the bulk and thus expect a smooth profile of bulk LDOS for an infinite number of samples. The large fluctuations are a specific property of our model, perhaps due to the presence of the magnetic field and the 3D nature. We have clarified this in Sec. 3 in the revised manuscript.



Comment 8: Given the current status of experimental research into the 3D quantum Hall effect, it could be of interest to consider whether there are any known materials that may be used for studying the case presented here, and if so how relevant parameters in those compare to the model Hamiltonians here.

Reply: We thank the referee for the question. The 3D quantum Hall effect can exist in amorphous systems because the bulk states are localized except at the critical point although the Landau levels broaden and overlap. This mechanism is generic, so the quantum Hall effect can exist in a large range of system parameters as long as the energy separation between the Landau levels is not too small. Therefore, it is quite promising to observe this phenomenon in experiments.

Following the referee's suggestion, we have tried the Dirac semimetal Cd_3As_2 in which the 3D quantum Hall effect in crystalline lattice have been experimentally observed [Nature 565 331 (2019)]. We use the tight-binding parameters of Cd_3As_2 given in PRL 119 136806 (2017) and generalize the tight-binding model to random lattices in the same way as in Sec. 2 of our manuscript except that here we add displacement $\delta\mathbf{r}$ obeying Gaussian distribution $D(\delta\mathbf{r}) = \exp\left[-\frac{|\delta\mathbf{r}|^2}{2\sigma^2}\right]/(2\pi\sigma^2)$ to the regular lattice. It is reasonable because there is residue spatial order in real amorphous materials. We see quantized Hall conductance up to the standard deviation $\sigma = 0.1$. For larger σ, i.e., stronger disorder, we cannot conclude whether the quantum Hall conductance still preserves under the system size within the capacity of numeric. To find concrete real material realizations of the 3D quantum Hall effect in amorphous systems is an important but highly nontrivial task, which deserves further efforts.

We have added a discussion on this in the conclusion part of the revised manuscript.

Comment 9: I do hold that there may be results here of sufficient novelty and relevance to the scientific community to warrant publication, especially with regards to the TR-invariant Hamiltonian. However, the comparison between the separate parts of the work and between this and previous works - chiefly [54] - should be elaborated on in the text. In its current format these aspects are left much too vague.

Reply: We thank the referee for judging our paper as having “sufficient novelty and relevance to the scientific community to warrant publication”.

We have followed the referee’s suggestion to further study the TR-invariant part which was less explored in the original version and make comparisons with the TR-symmetry broken case, as detailed in the reply to a previous comment and revised manuscript. We find both similarities and interesting points unique to the time-reversal symmetric case.

In the above, we have also clarified the difference of our paper from Ref. [54] (Ref. [56] in this version). The effect of the magnitude of the magnetic field has also been investigated. We thus believe that our work is now suitable for publication in SciPost Physics.

Comment 10: Add a discussion of to what extent results apply to case with / without time-reversal symmetry.

Reply: We thank the referee for the suggestion. As addressed in the reply above, in our revised manuscript, we have further studied the TR-invariant case which was less explored in the previous version. We find the bulk-boundary correspondence and mechanism of the localization of bulk states also apply for the TR-invariant case, as discussed in Sec. 5 of the revised version. The mechanism of the localization of bulk states is generic, so the effect can be realized in a large range of parameters and do not rely on the model details too much, as long as the Landau levels are not too close in the energy space.

We have added the discussion in Sec. 4, Sec. 5 and Sec. 6 in the revised manuscript.

Comment 11: Elaborate on the effects of the magnitude of B on the results.

Reply: We thank the referee for the suggestion. We have added a new Sec. 5 to elaborate on the effect of the magnitude of the magnetic field in the revised manuscript, as we have also addressed above.