Response to Ref 2

1. Referee: In Sec. 4.1, the authors seem to mix several setups. They apply the methods from Ref. [21], which studies the resolution of the entanglement entropy in a CFT on the complex plane with respect to a symmetry of the theory. Formula in Eq. (38) seems to assume a cylinder geometry. However, the final result (Eq. (40)) is compared with results from Ref. [24], where the entanglement asymmetry of an interval attached to a boundary that breaks the symmetry is studied. I am confused by the exact setup the authors are considering. This needs to be clarified.

   Response: In our paper, we have studied entanglement asymmetry of a CFT interval attached to a boundary of a strip for both in equilibrium and out-of-equilibrium settings. In both cases, we start with the state which manifestly breaks the symmetry generated by Q. By going to the upper-half plane (UHP), we find the appearance of the conformal boundary state which breaks the symmetry corresponding to the charged symmetry sector. Since a conformal boundary state can be written as a sum of primary and Virasoro descendent states, it is expected to break the symmetry of the full theory as well as that of individual charge sectors. That is why, the one-point functions in UHP becomes non-zero; they can be easily computed using standard method of images. In a cylinder, the ground state respect the symmetry as $[H, Q] = 0$ and $H|0\rangle = 0$. Hence in that case, we can not use the formal procedure of computing entanglement asymmetry which already assumed the fact that $[\rho, Q] \neq 0$. Thus if we wanted to work in the cylinder geometry, we would need to start from such excited primary state of the CFT. We hope that this clarifies our setting and we have added a brief discussion in the draft regarding this in the introduction section.

   We note that the structural similarity of (38) for a strip and that for the cylinder is well-known; it comes from the conformal mapping. This is also true in the same expression of entanglement entropy(upto an overall 1/2 factor) for an interval in full plane and an interval in an UHP attached to the boundary of half plane. The difference is that, in cylinder to complex plane mapping, primary one point function will be zero, while mapping from strip to the UHP, this is non vanishing. What we are computing is exactly composite twist one point function in an UHP, which, appealing to the method of images, is similar in structure to the two-point twist correlations in cylinder. We hope that this point addresses the issue raised by the referee.

2. Referee: The formulas in Eq. (37) are derived using the Appendix of Ref. [21], where the vertex opertors that implement the conserved charge $e^{-i\alpha Q}$ create a topological defect line. In that case, one can split the flux α among the replicas such that $\sum_{j=1}^{n} \alpha_j = \alpha$. This mimics the case of asymmetry in Eq. (33). However, the result I obtain for d_n is different from the expression in Eq. (37). I am obtaining $d_n = c(n - 1/n)/24 + \Delta \sum_{j=1}^{n} \alpha_j/n^2$. What is the crucial point I am missing to get Eq. (37) instead of the d_n I am obtaining?

   Response: In [21], the authors studied symmetry resolved entanglement entropy which is by definition

$$S_n(q) = \int d\alpha Tr(\rho_A^n e^{i\alpha(Q-q)}) = \int d\alpha Tr(\prod_{j=1}^{n} \rho_{A,j} e^{i\frac{\alpha}{n}(Q-q)}) \tag{1}$$

   Here, in the second equality we just split $\rho_A^n = \prod_{j=1}^{n} \rho_{A,j}$ such that $\rho_{A,j} = \rho_A$ for each j. This is possible since $[\rho_A, Q] = 0$ by definition. According to this line, we can think of computing n-sheeted partition function where in each sheet a vertex operator $e^{i\frac{\alpha}{n}(Q-q)}$ is threaded along the intervals. Along this line, Sela et al. has computed the symmetry resolved EE by computing the composite twist operator.

   In contrast, to compute entanglement asymmetry we have used the

$$Tr(\rho_{A,Q}^n) = \int \frac{d\alpha_1 d\alpha_2 \dots d\alpha_n}{(2\pi)^n} \mathrm{Tr}\left[\prod_{j=1}^{n} \rho_A e^{i\alpha_{j,j+1}Q}\right], \quad \alpha_{j,j+1} = \alpha_{j+1} - \alpha_j \tag{2}$$

   This is same as writing equation (33) in our draft where the integral variables are $\alpha'_j \equiv \alpha_{j,j+1}$, such that $\sum_{j'=1}^{n} \alpha_{j'} = 0$. Hence we have $(n-1)$ independent integrals to compute as from (33). We again use the composite twist formalism to compute n-sheeted partition function similar to [21] but with different α_j on each sheet. Due to different $\alpha_{j'}$, we will have a term in the twist dimension $\sum_{j'} \frac{\Delta(\alpha_{j'})}{n^2}$. If all $\alpha_{j'}$ are the same then it reduces to the Sela et al. (and referee's expression) as we explained above. Since we have only $(n-1)$ independent $\alpha_{j'}$ from (33), the sum over j' should run from 1 to $(n-1)$. This is the logic to obtain equation (37). We hope this will clarify referee's question (2).

3. Referee: I think there is a typo in Eq. (38). The last equality should be $Tr[\prod_{j=1}^{n} \rho_A e^{i\alpha Q}]$ instead of ρ_{QA}^n, which would be consistent with Eq. (39).

   Response: We have corrected this in the present version and we thank the referee for pointing out this typo.

4. Referee: I think Eq. (39) is only valid for Renyi index $n = 2$, but it is written for ΔS_n.

   Response: The expression in (39) is correct for any integer Renyi index $n > 1$. This is due to the fact that all the $(n-1)$ integrals have the same form(as they are being factorized due to the sum of j in d_n) and they can be written as a single integral to the power $(n-1)$. Taking log will merely cancel the prefactor $\frac{1}{n-1}$ of the Renyi entropy.

5. Referee: In the first sentence of Sec. 4.1, the authors write "The computation of ΔS_n in equilibrium has been carried out for cylinder geometry in several works [21, 22]". I would like to point out that, in Ref. [21], the entanglement asymmetry is not computed but rather the symmetry-resolved entanglement, which is the opposite situation. I would suggest to cite instead the paper JHEP 05(2024) 059 where the entanglement asymmetry is studied in the ground state of CFTs breaking a symmetry in the bulk and, in particular, in the Ising CFT. Ref. [25] also investigates asymmetry at equilibrium in CFTs in the complex plane.

   Response: We thank the referee for pointing this out and we have put the suggested citations in the new version of the draft.

6 Referee: Apart from the concerns above, I would like to ask the following question:

   vi) In the driven XY spin chain and PXP model, does the quantum Mpemba effect always occur when the symmetry is restored? It is not entirely clear to me from the discussion.

   Response: No it does not always occur and depends on the parameters chosen for the initial state. In the driven XY spin chain and in the PXP model Mpemba effect does not always occur when the symmetry is restored. We have chosen initial states in such a manner, that this effect occurs. There are other initial states as well which do not show Mpemba effect during the evolution of the entanglement asymmetry at the parameter points where the first order effective Floquet Hamiltonian gives rise to emergent U(1) conservation.

   Consider two initial states $|\phi_1\rangle$ and $|\phi_2\rangle$. For observation of the Mpemba effect, two things must happen simultaneously – (1) One of the initial states (say $|\phi_1\rangle$) must have a broader distribution of the conserved charge compared to the other state, which leads to $|\phi_1\rangle$ having more entanglement asymmetry at the initial time. (2) The state $|\phi_1\rangle$, must transport the charge through faster velocity modes compared to $|\phi_2\rangle$.

Now as time evolves entanglement asymmetry eventually decays as a result of the initial charge fluctuations surpassing across the boundary of the cut (used to define the entanglement entropy). If the two conditions mentioned above are satisfied simultaneously, then the state $|\phi_1\rangle$ will show a faster decay of entanglement asymmetry (compared to $|\phi_2\rangle$) even though it initially has a higher entanglement asymmetry, resulting in the observation of the Mpemba effect. Failure of simultaneous occurrence of both the conditions will result in a non-occurrence of the Mpemba effect. This is quite similar in a periodically driven or quenched system.

The quantitative analysis has been performed for the XY model in a quench scenario in Ref[13] of our manuscript] by considering charge transport through quasiparticles whose velocities are known analytically. Such an analysis can be generalized for the driven XY case considered in our manuscript, similar to the quasiparticle analysis performed to obtain the analytical picture in Fig1(a), in regimes, where the first order effective Floquet Hamiltonian gives rise to emergent magnetization conservation. This analysis is similar to that obtained in Ref 13. For the PXP case, the qualitative reasoning of the occurrence or non-occurrence of the Mpemba effect remains the same. However, as the PXP model is a non-integrable model, and a simple quasiparticles interpretation of excitations is not analytically tractable. Thus the search for initial states which satisfy the conditions mentioned above are in general difficult. For this one needs to look into charge transport properties of such systems, which, in our opinion requires a separate study.