Manuscript: "Theory of Order-Disorder Phase Transitions Induced by Fluctuations Based on Network Models"

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Dear Editors of SciPost Physics

I am deeply grateful for your handling of my manuscript.

Below is my response to the Referee2 's comments:

I sincerely appreciate your thorough review of my article and the valid concerns raised regarding its conclusions. First, I have clarified the ambiguities in the original text through concrete examples. Next, I provided a detailed introduction to the transformation process from lattice models to network models using specific instances and schematic illustrations. Following that, I offered an in-depth explanation of Equation (12), the core conclusion of this work. Furthermore, I rigorously demonstrated why the numerical simulation results(**0.32641871**) of the three-dimensional Ising model and those from the renormalization group approach serve as robust supporting evidence for my findings – specifically elucidating how these numerical results validate the critical exponent of 1/3 for the 3D Ising model. From the algorithmic perspective, I additionally addressed the inherent reliability of these conclusions through methodological validation. Finally, I have outlined the anticipated revision plan to address the raised concerns comprehensively.

1) For me the transformation from infinite lattice model to network model
is unclear.

The following sentence is obscure for various reasons :

"Firstly, all possible lattice sites in the lattice model are classified according to the magnitude of their interactions and the spin of the lattice sites themselves. Let Cij represent the weights of different types of lattice sites, where i denotes the spin type of the lattice site itself, and j represents different types of neighbor interactions. Then, different Cij values are treated as different network nodes. If there exists a transformation

relationship between different network nodes caused by fluctuations, the two network nodes are connected by a line segment. Finally, the different phases of the lattice model are labeled using the weights of different network nodes. If all lattice sites have spins pointing upwards, the weight of the corresponding network node is set to 1, while the weights of other network nodes are set to 0. This method does not focus on the specific position and momentum of any individual lattice site, but rather on the weights of different types of lattice sites. In other words, it attempts to capture the core physical information by using the weights and changes of different types of lattice sites."

here a series of doubts :

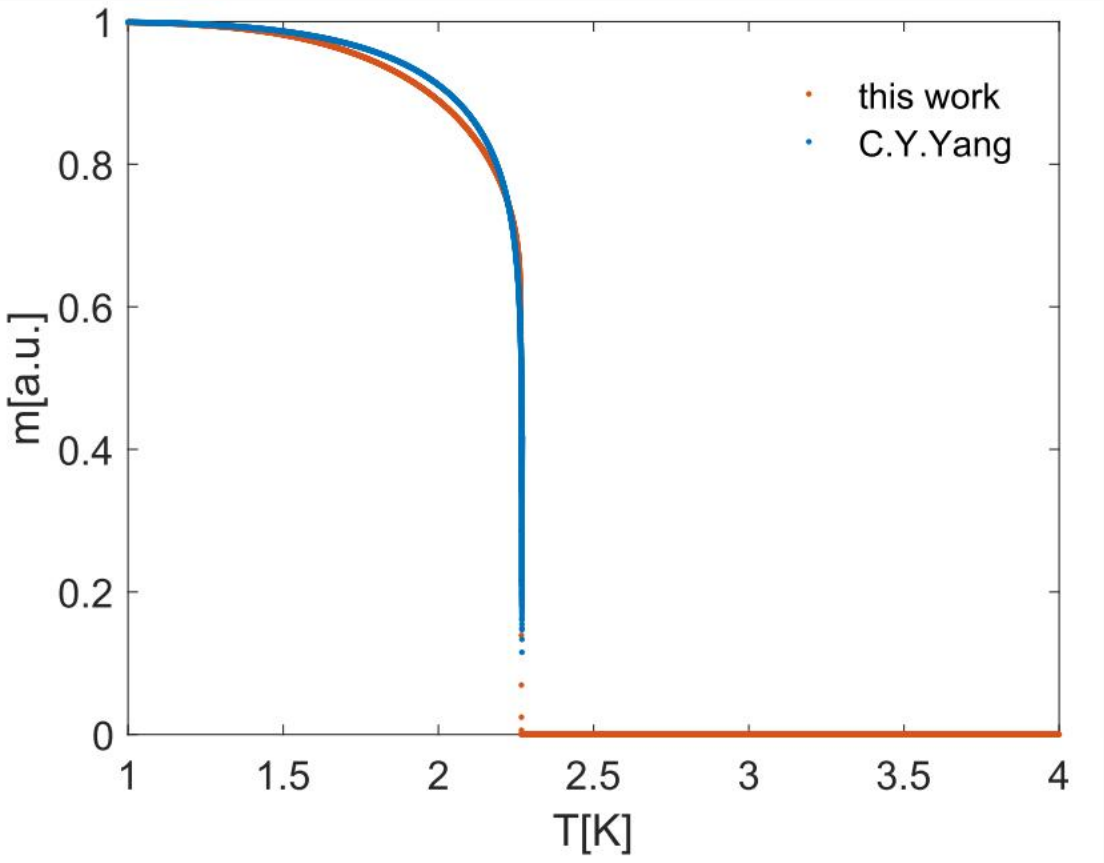
1i) when the author writes :
"lattice sites in the lattice model are classified according to the magnitude of their interactions and the spin of the lattice sites themselves." I guess that the author means "lattice sites are classified accordingly to the value of the spin variable occupying the considered lattice site and the value of the spins in the neighbouring lattices and of the values of their interactions" . However, I do not think there is an univoque mapping in such a case. I do not think a bi-univocal transformation can be constructed between a lattice and C_ij.

To illustrate, consider the case where only nearest neighbors are taken into account. In this scenario, each lattice point has the same total number of nearest neighbors, denoted as n Taking the two-dimensional Ising model as an example, each lattice point has 4 nearest neighbors, so n=4. When a lattice spin is oriented upward (as in the Ising model), we classify the lattice points based on the number of their nearest neighbors sharing the same spin direction. These different classes are then mapped to network nodes. Specifically, the values of identical neighboring spins range from a minimum of 0 to a maximum of n, resulting in n+1 distinct categories. For instance, in the 2D Ising model, the number of identical nearest neighbors ranges from 0 to 4, yielding 5 categories. The same classification applies to spins oriented downward. Since each lattice point in the Ising model can independently adopt one of two spin states (up or down), and the number of identical nearest neighbors is j, all possible lattice configurations are converted into corresponding network nodes. This method transforms the infinite two-dimensional Ising model into a network model with 10 nodes.

This transformation preserves the equivalence between lattice models and network models, as nodes in the network represent lattice points with identical interaction strengths—effectively reorganizing the lattice structure. However, reversing the process (from network to lattice) introduces multiple possibilities, which may correlate with entropy considerations. From a modeling perspective, reconstructing the Hamiltonian from the lattice perspective involves calculating interactions for each lattice point. For the ferromagnetic 2D Ising model, the calculated interaction values are $-4, -2, 0, 2$, and 4, which naturally form five distinct categories. By traversing all lattice points and dividing by

2 (to account for double-counting interactions), the derived expression matches the original Ising model Hamiltonian, confirming the rigor of this transformation.

Does this network representation retain critical information? The answer is yes. As demonstrated in my previous work, applying the mean-field approximation to nodes in the transformed network model allows precise derivation of the Magnetization versus temperature relationship for the 2D Ising model, as shown in the figure below.



1ii) This sentence is absolutely obscure :

" If there exists a transformation relationship between different network nodes caused by fluctuations, the two network nodes are connected by a line segment."

probably it means that depending on the temperature, if the energy allows for a flipping of two neighbours, then there is a connection among them ?

In this paper, the fluctuation-induced transition mechanism aligns with the treatment in Monte Carlo algorithms, where spin directions are directly altered without introducing additional variables. Specifically, the value of any lattice point's spin is modified—for example, transitioning from "spin-up" to "spin-down." Two types of transitions are considered here. The first occurs when the spin itself changes, leading to a shift in the network node it belongs to. The second arises when the lattice point itself remains unchanged, but its neighboring lattice points undergo variations, thereby altering the

network node to which the original lattice point belongs. By connecting all possible transitions between network nodes resulting from such spin flips with line segments, the relationships between nodes are explicitly mapped.

2) The author should present in a simple case (Ising model ?) the transformations he has in mind to pass from lattice to network, step by step.

This work presents a substantially different theoretical approach compared to conventional methods, dedicating significant attention to explicating the foundational theory. The following sections detail the network model construction through concrete examples and schematic illustrations.

Network Model Construction

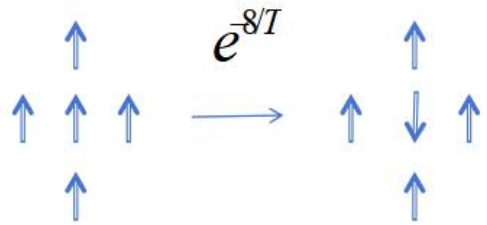
Example: 2D Ising Model with Nearest-Neighbor Interactions
Consider a 2D Ising model where each lattice site exhibits spin-up/down states. Sites are classified based on their own spin state and the number of nearest-neighbor sites (4 neighbors) sharing the same spin:

- **Spin-up classification**: 5 categories (0-4 matching neighbors)
- **Spin-down classification**: 5 categories (0-4 matching neighbors)

This results in 10 distinct classes (C_{15}-C_{25}). Mathematically, for the Hamiltonian $H=1/2\sum_{<i,j>} S_i S_j$, I reorganize terms by these classes. The factor of 1/2 accounts for double-counting interactions. Importantly, this classification covers *all possible configurations* in the infinite 2D Ising model through 10 network nodes, where node weights represent configuration probabilities.

State Transitions and Network Dynamics

Case 1: Spin Flip of Central Site



Initial State: Central site has spin-up with 4 matching neighbors (Class C_{15}).

Active Transformation: Flipping the central site changes its state to spin-down with 0 matches (Class C_{21}).

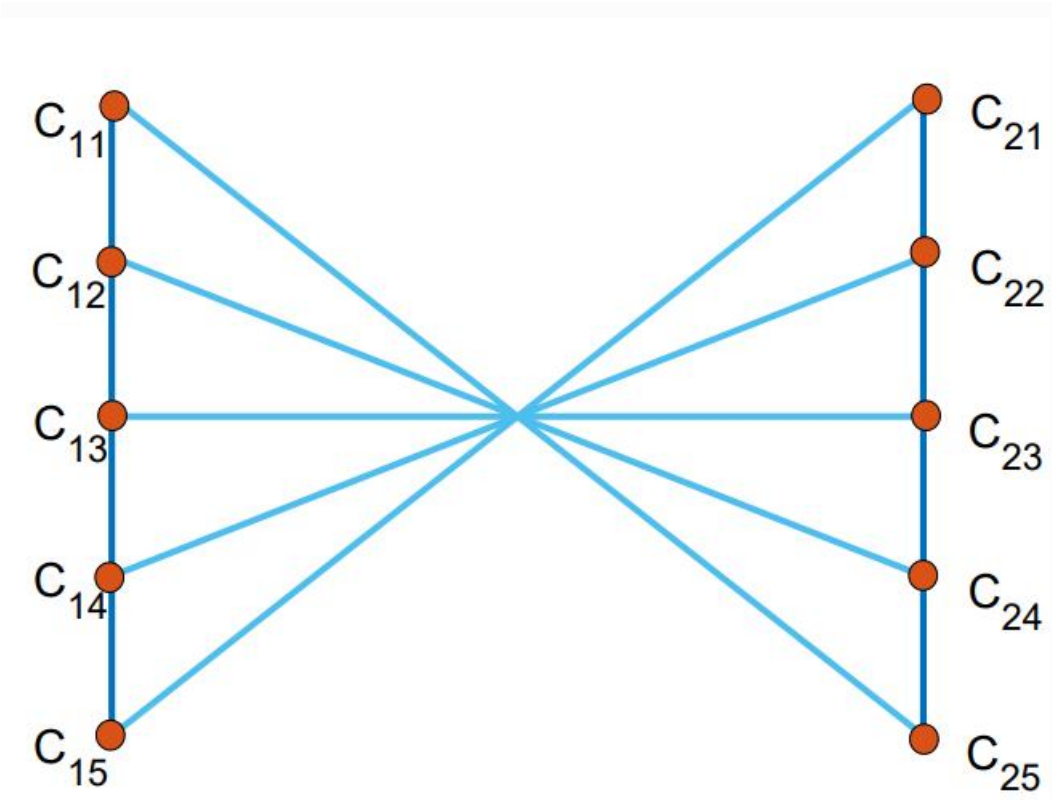
Passive Transformation: This flip simultaneously alters neighboring sites' classes from $C_{15} \rightarrow C_{14}$ (each neighbor loses one match).

Case 2: Neighbor Spin Flip

Initial State: Central site remains spin-up with 4 matches (C_{15}).

Passive Transformation: Flipping a neighboring site reduces its match count to 3 (Class C_{14} for the neighbor), indirectly modifying the central site's class to C_{14}.

These two cases illustrate all possible transformations under nearest-neighbor interactions. The complete network structure emerges from considering all such transitions.



Network Node Labeling Convention

Nodes are labeled Cij :

$i \in \{1,2\}$: Spin state (1=up, 2=down)

$j \in \{1,5\}$: Number of matching neighbors (1=0 matches, 5=4 matches)

Example:

C_{15} : Spin-up with 4 matches

C_{13} : Spin-down with 2 matches

Generalization to Higher Dimensions

3D Ising Model: Classifies sites into 14 nodes (7 match counts × 2 spins)

N-dimensional Models: Follows similar classification logic with $2(N+1)$ nodes

This framework applies to various lattice models through analogous interaction-based classifications.

Active vs. Passive Transformations

Transformation Type	Definition	Network Impact

Transformation Type	Definition	Network Impact
Active	Direct spin flip of target site	Horizontal transition between nodes
Passive	Indirect flip via neighbor changes	Vertical transition within nodes

Key Insight: A single active transformation (e.g., flipping site A) corresponds to four simultaneous passive transformations (its four neighbors' state changes).

Deriving High-Order Detailed Balance
Unlike Monte Carlo simulations that use active transformations, this work focuses on passive transformations to establish:

Microstate Transition Probabilities: Calculate joint probabilities for four-site passive transformations

Balance Equations: Derive relationships between node weights during phase transitions

Phase Transition Analysis Using Waterfall Metaphor

Initial State (T=0): All nodes in C_{15} with weight 1

Slow Phase ($C_{15} \rightarrow C_{13}$): Gradual weight migration resembling water approaching a cliff edge

Rapid Phase ($C_{13} \rightarrow C_{23}$): Abrupt weight redistribution analogous to water cascading

This demonstrates how passive transformations effectively capture critical transition dynamics missed by traditional active-only approaches.

3) I do not understand large part of what written by the author neither his logic, however Eq (12) I guess is the magnetization per spin as a function of temperature in a Ising Model (by the way the temperature T has been not introuced as well as the magnetization m), for 2 dimensions the solution of this problem has been found by Onsager :

<m> = [1 - 1/sinh(2J/KbT)^4]^1/8

this would mean that in 2 dimensions, by assuming J/Kb =1, from eq 12 one gets the 2 dimensional results for n=4 and k=2, why ? What does it mean ?

In this paper, the treatment of temperature aligns with Monte Carlo algorithms, where temperature influences transition probabilities between network nodes via the detailed balance principle. For thermodynamic phase transitions, these probabilities can be rigorously derived. The transition probabilities may be directly determined by temperature or other variables (e.g., in quantum phase transitions). Similarly, the calculation of magnetization follows the same approach as in Monte Carlo methods. Taking the Ising model as an example, the magnetic susceptibility is computed by subtracting the weight of spin-down states from that of spin-up states. Although Equation (12) is formulated for broader scenarios and does not explicitly incorporate temperature T, the transition probabilities remain derivable from temperature variations.

Here, n represents the number of nearest neighbor lattice points. For the two-dimensional Ising model, each lattice point has 4 nearest neighbors. This means that flipping a single lattice point in the 2D Ising model alters the states of its four nearest neighbors. Meanwhile, k denotes the minimum number of boundary lattice points required to combine and form a maximum-entropy lattice point.

Taking the ferromagnetic Ising model as an example, at absolute zero temperature (T=0), one network node carries a weight of 1, while all other nodes have a weight of 0. This configuration is termed the single-node structure . After the phase transition, the fully disordered system can be directly calculated using temperature and the principle of maximum entropy. This disordered configuration is referred to as the maximum-entropy structure .

A boundary exists between the single-node structure and the maximum-entropy structure. A direct transition without passing through this boundary would imply no phase transition occurs, so this intermediate boundary structure is inevitable.

The passive transformation from the single-node structure to the maximum-entropy structure proceeds via the boundary structure. Flipping a node in the single-node structure generates n boundary structure nodes. Conversely, each boundary structure node can produce two maximum-entropy structures through spin flips. The sole driving force for this transformation is the flipping of lattice points.

For the Ising model, n corresponds to the number of nearest neighbors, which is twice the value in the original Ising model. Specifically, flipping a node in the single-node structure produces n boundary structure nodes. The parameter k, on the other hand, represents the minimum number of boundary lattice points required to transition from a boundary

structure to a maximum-entropy structure's central node via flips. The values of k can be directly calculated for different dimensions. Both n and k are rigorous, intrinsic properties of Ising models in their respective dimensions.

4) Furthermore, for what I understand from Eq (12) and Fig 2 the author has analytically solved the problem of Ising in 3, 4 , and 5 dimensions, and found the corresponding critical exponent beta, at least for the magnetization at the critical point.

As far as I know the Ising model in 3d has not yet been solved, see

Ferrenberg, Alan M., Jiahao Xu, and David P. Landau. "Pushing the limits of Monte Carlo simulations for the three-dimensional Ising model."
Physical Review E 97.4 (2018): 043301.

and also more recently :

Liu, Zihua, et al. "Critical dynamical behavior of the Ising model." Physical Review E 108.3 (2023): 034118.

How the results of the author do relate to the above literature ?

From wikipedia the best estimation (numerical) of the Beta exponent for the 3D Ising model is 0.32641871(75) and not 1/3 as reported by the authors.

The three-dimensional Ising model exhibits fractal phenomena, which emerge precisely at the phase transition critical point. Thus, the formation of fractal structures precedes the onset of phase transition in this model. However, this observation conflicts with the use of periodic boundary conditions (PBCs) in Monte Carlo (MC) simulations and renormalization group (RG) analyses. Under PBCs, the system cannot form fractal clusters exceeding the periodic boundary length, effectively making such simulations a truncated approximation. If the existence of fractal structures in the three-dimensional Ising model remains debated, it necessarily implies the presence of clusters that exceed simulated system sizes. In summary, the critical phenomena of the three-dimensional Ising model in the thermodynamic limit (infinite system size) remain unknown and are evidently distinct from results obtained under periodic boundary conditions (PBCs). Phase transitions, by definition, inherently describe behaviors in the thermodynamic (infinite-size) limit. Experimental results, meanwhile, are inevitably influenced by finite-size effects and physical boundaries.

Extensive evidence confirms the high efficiency and accuracy of MC and RG methods. Therefore, it is highly probable that the analytical solution (if rigorously derived) would

align closely with both MC/RG numerical results and experimental data. The discrepancy between 1/3 and the widely accepted numerical value 0.32641871(75)—a difference of 0.0069—remains within acceptable tolerance. All three approaches (MC, RG, and experiments) yield consistent results but are inherently affected by finite-size truncation (via PBCs) and boundary effects. Consequently, the analytical solution is expected to slightly exceed these values.

The MC and RG results provide robust numerical support for the critical exponent derived in this work. In contrast, Landau's mean-field theory predicts $\beta=0.5$, while our proposed value of 1/3 demonstrably aligns more closely with the essence of phase transition physics. This conclusion underscores the fluctuation-driven phase transition theory central to this paper, where critical phenomena arise from collective spin fluctuations rather than mean-field approximations.

From another perspective, all results in this work are derived directly from Equation (12), with minimal assumptions. The single-node structure at $T=0$ and the maximum-entropy structure after the phase transition are objective physical entities. If no boundary structure existed between these two states, a direct transition would occur, eliminating the phase transition point—as seen in the one-dimensional Ising model. Hence, boundary structures are inevitable. Furthermore, this study focuses solely on the impact of fluctuations (specifically modeled as spin deflections in this work) on the results. While I believe the critical exponent for the three-dimensional Ising model presented here is an analytical solution, I cannot rigorously claim this without formal mathematical proof. Nevertheless, Equation (12) is consistent with all existing findings: the absence of a phase transition in 1D, the exact solution in 2D, numerical results in 3D, and critical exponents in higher dimensions ($d\geq4$). This universality underscores the value of Equation (12).

Manuscript Revision Plan

In the Conclusion section, I will provide a detailed discussion on the current research progress regarding the critical exponent β of the three-dimensional Ising model, encompassing results from Monte Carlo (MC) simulations, renormalization group (RG) analyses, and experimental investigations.

1)**Reference Integration**

Systematically incorporate all cited works into:

Introduction (contextual framing)

Conclusion (theoretical implications and future directions)

Ensure seamless integration with existing narrative flow

2)**Methodological Appendix**

Transfer detailed model construction procedures and case study demonstrations to:

Appendix C:

- Comprehensive derivation of network model formalism
- Step-by-step validation with Ising model examples
- Comparative analysis with Monte Carlo simulations

Include:

- Schematic diagrams illustrating transformation pathways
- Tabular summaries of critical exponent comparisons

3)**Structural Optimization**

Streamline main text by:

- Removing redundant technical explanations
- Concentrating core innovations in theoretical framework
- Reserving experimental validations for dedicated sections

Enhance reader navigation through:

- Updated table of contents
- Cross-referencing between main text and appendices
- Strategically placed summary paragraphs

Sincerely,
Yonglong Ding