# Response to Report 3 on "Four no-go theorems on the existence of spin and orbital angular momentum of massless bosons" (scipost_202412_00011v2)

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Dear Referee,

We sincerely thank you for your time in reviewing our manuscript and providing useful feedback on our work. You point out that it would strengthen our paper to more directly address the question of to what degree the helicity operator can act as a replacement of the SAM for massless particles (relaxing definitions as needed). We address this question in detail below and have added a similar discussion of this to end of Section 5, beginning in the last paragraph on page 11. We hope that you find our revisions satisfactory.

**Referee comment**:

> This manuscript systematically investigates the fundamental limitations in decomposing the total angular momentum of massless bosons into orbital and spin components. The core framework is established through equations (1) and (15)-(16), anchored by the canonical angular momentum commutation relation:
>
> $$[J_a, J_b] = i\epsilon_{abc}J_c. \tag{1}$$
>
> The authors have further explored generalized scenarios involving non-internal symmetry implementations for spin operators and non-SO(3) symmetry extensions for both spin and orbital angular momentum operators. Through these extensions, they consistently arrive at the central conclusion that such decomposition remains fundamentally unattainable. The study's original motivation rooted in experimental optics contexts is valuable, offering insights into angular momentum decomposition challenges within specific physical implementations.
>
> While the conclusion is physically sound, the analysis could be significantly strengthened by addressing the unique ISO(2) little group structure to massless particles. Specifically, the critical distinction between helicity operators ($h$) and spin angular momentum operators $\boldsymbol{S}$ warrants deeper consideration. A potentially fruitful modification might involve redefining the commutation relations in (15)-(16) as:
>
> $$[J_a, h] = \cdots, \ [h, h] = 0 \tag{2}$$

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while relaxing constraints on orbital angular momentum. This proposed refinement does not diminish the current work's significance, but rather complements it by addressing the fundamental differences between massive and massless particle representations.

**Reply:**

As you point out, the distinction between spin and helicity operators is essential for understanding the differences between massive and massless particles. These operators are analogs of each other in the sense that both result from Wigner's little group method. In particular, the spin operator $\boldsymbol{S}^m$ is the generator of the action of the massive little group SO(3). In the massless case, the little group is instead ISO(2) (and in terms of the representation theory, it really behaves like SO(2) since the inhomogeneous terms act trivially for all particles in the standard model [1]). The helicity operator $\chi \doteq \boldsymbol{J} \cdot \hat{\boldsymbol{k}}$ is the generator the SO(2) little group (note we write $\chi$ instead of $h$ for the helicity operator here to be consistent with the manuscript). As the spin and helicity both originate from the little group method, it is natural to ask to what degree the helicity operator replaces the role of the spin in the theory of massless particles. An answer to this question is scattered throughout the current manuscript and our paper on photon topology [2]. However, you rightfully point out that it is of interest to consolidate these results to give a direct answer to this question. This is described below and we have added a similar discussion to the end of Section 5, beginning in the last paragraph on p. 11.

The goal of Wigner's little group method [1–3] is to classify the unitary irreducible representations (UIRs) the Poincare group, as these correspond to particles. Wigner showed that it is sufficient to classify the representations of the little group, that is, to classify the representation described by $\boldsymbol{S^m}$ for massive particles and the helicity $\chi$ for massless particles. From the representation theories of SO(3) and SO(2), the massive UIRs are classified by the eigenvalues of $(S^m)^2 = \boldsymbol{S}^m \cdot \boldsymbol{S}^m$ while the massless representations are classified by the eigenvalues of $\chi$. This classification essentially works because $(S^m)^2$ and $\chi$ commute with all generators of the Poincaré group. That is, in the massless case (see e.g. [2], Thm. 33)

$$[J_a, \chi] = 0 \tag{3}$$

$$[k_a, \chi] = 0 \tag{4}$$

$$[H, \chi] = 0 \tag{5}$$

$$[K_a, \chi] = 0. \tag{6}$$

where $H$ is the Hamiltonian and $\boldsymbol{K}$ is the generator of boosts. In this sense, spin and helicity play an analogous role, but even here we see the difference that massive particles are classified by the eigenvalues of the operator $(S^m)^2$ rather than $\boldsymbol{S}^m$.

The analogy of $\boldsymbol{S}^m$ and $\chi$ shows up in the context of Wigner's little group method, but this theory does not *a priori* say anything about SAM-OAM decompositions. That such a decomposition results for massive particles is due to the fact that the little group happens to be SO(3), which is associated with angular momentum operators. The theory of SAM and OAM for massive particles is immediate since one has the angular momentum operator $\boldsymbol{S}^m$ generating the little group symmetry and the total angular momentum operator $\boldsymbol{J}$ from the canonical copy of SO(3) in the Poincaré group describing spatial rotations. One can then define $\boldsymbol{L}^m = \boldsymbol{J} - \boldsymbol{S}^m$.

However, in the massless case, $\chi$ generates a representation of SO(2), and this gives no immediate SAM-OAM splitting. The commutation relations for $\chi$ in Eq. (3) are highly useful in classifying massless particles, but they are quite different than those satisfied by $\boldsymbol{S}^m$. There is no obvious theoretical way in which $\chi$ replaces $\boldsymbol{S}^m$ as an angular momentum. One of the differences of $\chi$ and $\boldsymbol{S}^m$ is that $\chi$ is a scalar operator. There have been attempts [4] to "vectorize" the helicity operator into $\boldsymbol{J}_\parallel = \chi\hat{\boldsymbol{k}} = (\boldsymbol{J} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}}$, and then one can define $\boldsymbol{J}_\perp = \boldsymbol{J} - \boldsymbol{J}_\parallel$. There are two major drawbacks of this, the first being that $\boldsymbol{J}_\parallel$ and $\boldsymbol{J}_\perp$ are still not angular momentum operators in that they do not satisfy SO(3) commutation relations. The second is that this is an ad hoc procedure. While it is very clear in Wigner's little group why the helicity operator $\chi$ is of interest, it is not clear what the vectorized operator $\chi\hat{\boldsymbol{k}}$ really is, especially since we know it is not a generator of a rotational symmetry. However, it is interesting to note that $\boldsymbol{J}_\perp$ and $\boldsymbol{J}_\parallel$ are gauge-invariant and thus measurable. Related to this, they are constructed entirely in terms of Poincaré generators so they do not necessarily involve some arbitrary choice (such as a choice of gauge). Thus, while the nature of $\boldsymbol{J}_\perp$ and $\boldsymbol{J}_\parallel$ are not currently well-understood, it is possible that there is some theory which explains the origins of this splitting. Ideally, it would result from some general construction for representations of the Poincaré group and in the massive case it would give the standard SAM-OAM decomposition. We are actively exploring this research avenue.


[1] S. Weinberg, *The Quantum Theory of Fields*, Vol. 1 (Cambridge University Press, 1995).

[2] E. Palmerduca and H. Qin, Photon topology, Phys. Rev. D **109**, 085005 (2024).

[3] E. Wigner, On unitary representations of the inhomogeneous lorentz group, Annals of Mathematics **40**, 149 (1939).

[4] I. Bialynicki-Birula and Z. Bialynicka-Birula, Canonical separation of angular momentum of light into its orbital and spin parts, Journal of Optics **13**, 064014 (2011).