Manuscript: "Theory of Order-Disorder Phase Transitions Induced by Fluctuations Based on Network Models"

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Dear Editors of SciPost Physics

I am deeply grateful for your handling of my manuscript.

Below is my response to the Referee3 's comments:

I sincerely appreciate your review of my manuscript. To address the concerns, I will first elaborate on the reasoning process of the entire paper using the three-dimensional Ising model—which has the most contentious results—as a representative example. I will provide precise definitions of relevant concepts specifically within the context of the transformed network model. Subsequently, I will demonstrate how the corresponding conclusions can be obtained while circumventing the direct application of the maximum entropy principle and the principle of least action as previously described in the text. I apologize for the omission of Onsager's seminal work; its inclusion is undoubtedly essential and will be rectified. In response to your comments, I will comprehensively revise the manuscript, systematically replacing derivative symbols with representations specific to the transformation framework to enhance clarity.

First, the lattice model is transformed into a network model. For the three-dimensional Ising model, the number of nearest neighbors per lattice site is 6 (denoted as n in the text). For a spin-up lattice site, the number of its nearest neighbors sharing the same spin can range from 0 to 6, resulting in **7 distinct categories**. Similarly, for a spin-down lattice site, the same classification applies. This results in a total of **14 categories** for all lattice sites. These 14 categories are then mapped to **14 corresponding network nodes**, denoted by the symbol Cij, where:

- i represents the spin state of the lattice site itself. In the Ising model, spin can only take two values: **1** for spin-up and **2** for spin-down.
- j represents the strength of nearest-neighbor interactions, with **7 possible values** (1 through 7). Here, $j=1$ corresponds to cases where **0** nearest neighbors share the same spin as the central site, $j=2$ corresponds to **1** matching neighbor, and so on, up to $j=7$, which represents **6** matching neighbors.

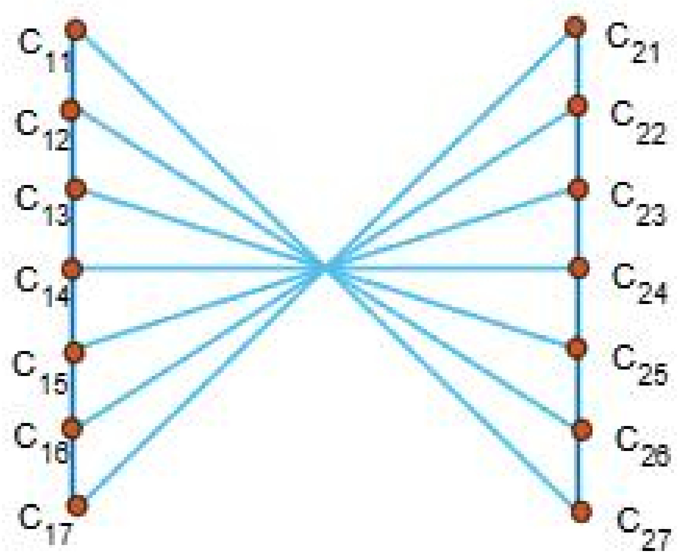
Now consider the flipping of a lattice site in the three-dimensional Ising model. According to the classification above, flipping a lattice site alters its category. For example, if a spin-up lattice site (with all 6 nearest neighbors sharing the same spin) is flipped, it becomes a spin-down site, and all its nearest neighbors now differ from it. This type of transformation, induced by the **spin flip of the lattice site itself**, is termed an **active transformation** in this work.

Simultaneously, when the central lattice site flips, the categories of its six neighboring sites also change—though the neighboring sites themselves remain unchanged. This occurs because the central site is a nearest neighbor to these six sites. The alteration of the central site's state directly impacts the categories of its neighbors. This indirect transformation caused by the central site's flip is referred to as a **passive transformation**.

All possible transformations are connected via edges in the network. For instance, a transition from C17 to C21 can occur:

- C17 corresponds to a **spin-up lattice site** with **all 6 nearest neighbors sharing the same spin**.
- After flipping, the site becomes **spin-down**, and all its neighbors now **differ** from it, corresponding to C21.

By mapping all such transformations, the network structure is constructed as follows.



This network structure rigorously encompasses all possible lattice site types and transformation relationships. The weights of distinct network nodes represent the relative prevalence of each lattice site category within the original lattice model, with the total weight across all nodes summing to 1. Consequently, this network model is applicable to infinite systems, enabling its use for studying the infinite three-dimensional Ising model.

How does the ferromagnetic phase transition of the Ising model manifest in this network structure?

For the ferromagnetic Ising model at zero temperature, all lattice sites align uniformly either spin-up or spin-down (spin-up is chosen as the example here). This uniform alignment corresponds to **C17 having a weight of 1**, while all other network nodes have zero weight. After the phase transition, the weights of nodes in the **first and second columns of the network become equal, each contributing 1/2 to the total weight** (note: this work assumes that post-transition node weight distributions follow temperature-dependent random distributions, which can be directly computed as they are independent of the primary analysis framework; further details are omitted here).

Next, analogous to Monte Carlo methods, the transition probabilities between adjacent columns of network nodes can be calculated using detailed balance. For example, consider a spin-up lattice site (C17) where all six nearest neighbors are also spin-up. Flipping this site transitions it to C21, and vice versa. The detailed balance formula directly yields the **weight ratio** between C21 and C17. This calculation applies **only to transitions between nodes in adjacent columns** (e.g., C17↔C21), not to transitions within the same column. This treatment aligns entirely with the Monte Carlo approach.

In the Ising model, flipping a single lattice site induces changes that can be represented by edges in the network. For instance, C17↔C21describes a **successful spin flip** via an **active transformation**. Simultaneously, the same physical flip can be interpreted as a **passive transformation**: when the central site flips, its six neighboring sites *passively* transition from C17 to C16 (since their shared spin alignment with the central site changes). Both descriptions correspond to the *same physical flip*, but differ in focus:

- **Active transformation**: The central site itself changes state (C17↔C21).
- **Passive transformation**: Six neighboring sites change state (C17↔C16).

Crucially, the number of passive transformations triggered by a single flip is n-fold greater than the active transformation, where n is the number of nearest neighbors (6 in 3D). This reflects the combinatorial impact of a single spin flip on its surrounding lattice sites.

Next, I investigate the phase transition based on these distinctions. For the three-dimensional Ising model, there are two critical types of network nodes:

1. **C17**: At zero temperature, all lattice sites are spin-up, corresponding to C17 having a weight of 1, while all other nodes have zero weight.
2. **C14 and C24**: By classification rules, these nodes represent configurations where the number of spin-up and spin-down nearest neighbors are equal. Using the detailed balance equation, the weights of C14 and C24 are found to be equal, making them **central nodes**. These three node types form the basis for analyzing phase transitions.

Key Insight: Passive transformations—not active transformations—dominate the phase transition dynamics. For example, the direct weight flow from C17 to C21 (via active transformations) is negligible at high temperatures. Instead, the transition manifests through cascading weight flows involving passive transformations:

- As temperature increases, weight flows from C17 to C16, then splits into C15 and C14.
- At C14, half the weight transfers to C24, triggering symmetry breaking and the phase transition.

Metaphorical Explanation: Imagine a flock of sheep attempting to cross a river. Most sheep cannot ford the deep channel directly (analogous to C17⟶C21) but instead follow the shallow banks (represented by C16-mediated pathways). As temperature rises, the "flow" of sheep shifts from deep to shallow routes, culminating in a split at the critical point (C14⟷C24).

Role of Boundary Structure C16:
The transition relies on C16 acting as a **boundary structure** that mediates between stable states. Unlike high-dimensional Ising models where direct state conversion is inefficient, C16 persists as a transient hub connecting higher-layer nodes (e.g., C17) to lower-layer nodes (e.g., C15). Its position—adjacent to both stable and critical nodes—enables it to regulate weight redistribution during phase transitions. The necessity of C16 for finite-size effects and its structural linkage to base network nodes will be elaborated further below.

The flipping of lattice sites is a **stochastic process**: throughout the system, sites continuously transition from C17 to C16, while a large number simultaneously transition back from C16 to C17, maintaining equilibrium. This equilibrium applies to all network node transitions. C16 is classified as a **boundary structure** because, after sites transition en masse from C17 to C16, they are more likely to revert to C17 or remain in C16 rather than transitioning to C15 (temporarily ignoring active transformations). Below, we explain why C16 preferentially reverts to C17 rather than ultimately transitioning to C14.

Method to Determine C16's Conversion Preference:
By definition, a C16 node corresponds to a **spin-up lattice site** with five spin-up nearest neighbors and one spin-down neighbor. If half of its neighboring nodes are C17 (spin-aligned) and the other half are C14 or C24 (spin-mismatched), the probabilities of C16 transitioning to these two categories can be directly calculated. The category with the higher probability indicates C16's dominant transition tendency.

Case 1: Extremely Low Temperature
When the temperature is near absolute zero, randomly selected C17 nodes do not flip. Let:

- $q1$ Probability of C16 transitioning to C17.
- $q2$ Probability of C16 transitioning to C14 or C24.

At ultralow temperatures, $q1$ dominates due to the energetic preference for spin alignment. This reflects the system's rigidity near zero temperature, where deviations (e.g., C16) are transient and resolve quickly.

Following the aforementioned method for determining transition probabilities, a C16 node has **three C17 neighbors**, **two C14 neighbors**, and **one C24 neighbor**. At low temperatures, when randomly selecting neighbors for flipping:

- Neighbors in C14 or C24 will flip when selected.
- Neighbors in C17 will **not** flip.

First Flip:

- Probability of transitioning C16⟶C17: 1/6(selecting one C17 neighbor out of six total).
- Probability of transitioning C16⟶C15: 1/3 (selecting one of the two C14 neighbors, which then flip to C15).

Second Flip (if C16⟶C15 occurs):

- From C15, transitioning back to C16 has a probability of 1/3.
- Transitioning to C14 has a probability of 1/6.

This results in the equilibrium equations:

$1/6+1/3 * 1/3 *q1=q1$ (for C16 ↔C17)
$1/3*1/6+1/3*1/3*q2=q2$ (for C16 ↔C14 or C24)

Solving these gives:

- q1 =3/16 ,q2 =1/16

Thus, C16 exhibits a stronger tendency to revert to C17 (q1 >q2) under low temperatures. As temperature increases, q1 gradually decreases while q2 correspondingly increases. Given that C16 is the closest boundary node to C17 and the focus here is on critical behavior, selecting C16 as the sole boundary node is justified.

Simplified Weight Flow:

1. Weight flows from C17 to C16 .
2. At C16 , most weight is retained or returns to C17 (blocking majority of transitions).
3. A small fraction flows to C14 , which then splits equally to C24 via passive transformations.

This establishes a dynamic equilibrium:

1. C17 and C16 maintain a balanced exchange.
2. C16 and C14 also balance their interactions, enabling the system to model critical phenomena.

Next, I derive this equilibrium relationship using **higher-order detailed balance**. To illustrate, consider a three-dimensional Ising model with all spins aligned upward. Flipping a single lattice site transforms it from C17 to C21 , while its six nearest neighbors transition from C17 to C16 . At ultralow temperatures (where only C17 , C21 , and C16 exist), the number of C16 nodes becomes **six times** the number of C21 nodes. This factor of 6 corresponds to the number of nearest neighbors (n). Here, **active transformations** obey standard detailed balance, while **passive transformations** follow higher-order detailed balance.

Efficiency Principle Under Detailed Balance:

1.

Flipping a C17 Node:

2.

- If all neighboring nodes are C17, flipping C17 converts **six neighboring nodes** from C17 to C16.
- This results in **six upward weight flows** (from C17 to C16) in the same column.

3.

Flipping a C16 Node:

4.

- A C16 node (spin-up with five C17 neighbors and one C16 neighbor) flips to C21, converting **five neighbors upward** (to C16) and **one neighbor downward** (to C15).

Combined Effect:

- Flipping **one C17** and **one C16** leads to:
  - **Six upward flows** (from C17 →C16).
  - **Four net upward flows** (five from C16 →C16 neighbors, one from C16 →C15).
- This is equivalent to **six upward flows from C17** and **four upward flows from C16**, demonstrating that passive transformations can be treated as pseudo-active transformations of the same node type.

Key Insight:

- **C17 flips** drive **six upward weight flows**.
- **C16 flips** effectively drive **four upward flows** (net of five upward and one downward).
  This efficiency principle allows the system to model critical behavior by prioritizing C16 as the boundary node, where most weight remains trapped until phase transition temperatures are reached.

Next, I rigorously define k, which arises from the framework's constraints. In this work, lattice site transformations are exclusively modeled via spin flips, with no additional mechanisms considered. As established earlier, a transition from C17 to C16 occurs when a spin-up site flips, while C16 transitions to C14 under specific conditions.

Derivation of k:

- Flipping a single C17 node generates **six C16 nodes** (one per nearest neighbor).
- However, generating a C14 node requires **two(k) simultaneous C16 flips**:
  - Each C16 flip produces one C14 node on a*verage* (due to passive transformations).
  - Thus, flipping **one C17** indirectly leads to **three C14 nodes** (since six C16 nodes are created, and each contributes a C14 node with probability 1/2).
    This results in a **critical exponent β=1/3** for the three-dimensional Ising model.

Dimensional Generalization:

- **2D Ising model**: Four lattice sites transition to boundary nodes, each producing two central nodes (**β**=1/8).

- **4D+ models**: Accounting for integer rounding in neighbor counts, **β**=1/2.
  These values are mathematically exact within the framework. Notably, transitioning to C14 implies that **half the weight flows to C24** (via passive transformations).

Phase Transition Formula:
Using these principles, the critical behavior of the three-dimensional Ising model is derived as:

Critical exponent **β**=1/3

This result aligns with the hierarchical weight redistribution mechanism and the boundary node dynamics described above.